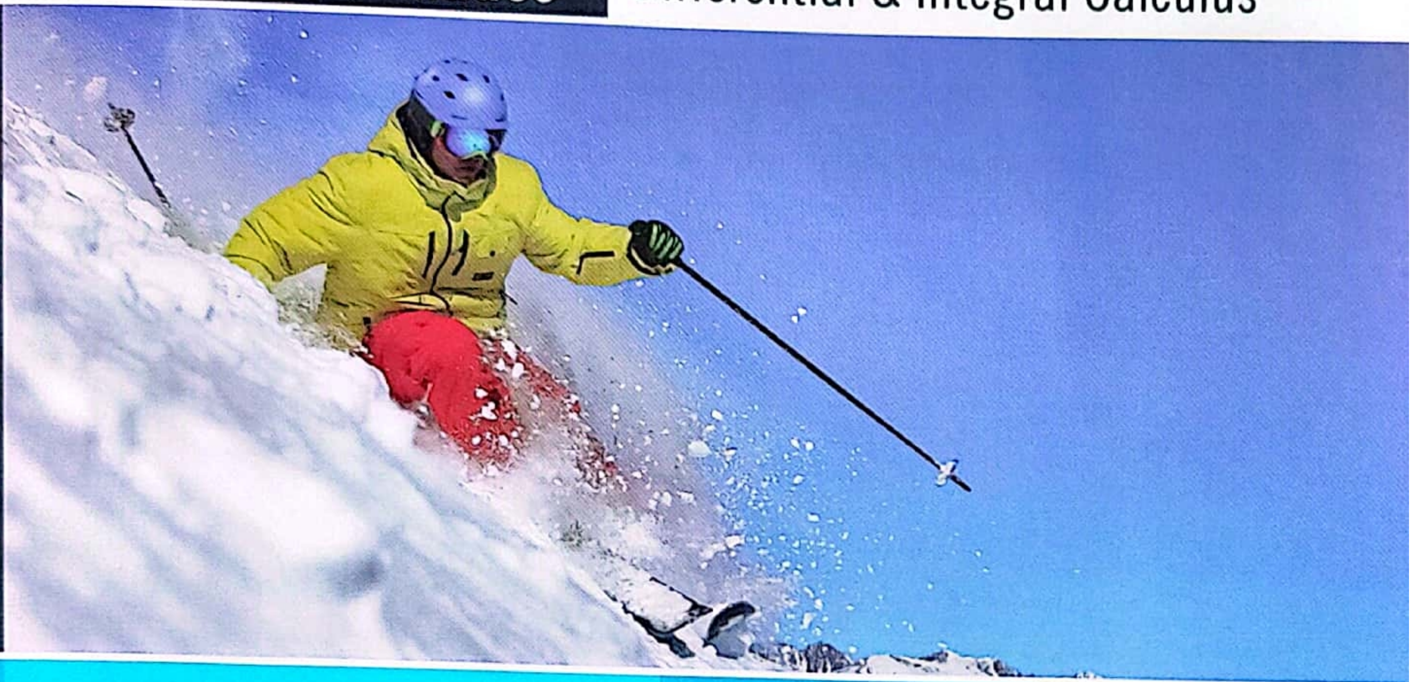
Pure Mathematics

Differential & Integral Calculus



GUIDE ANSWERS

Question Bank
& Practice Exams

EL-MOASSER

By a group of supervisors

3rd Sec

2022

EL-MOASSER

By a group of supervisors

PURE MATHEMATICS

Differential & Integral Calculus

Question Bank
& Practice Exams
Guide Answers

GPS

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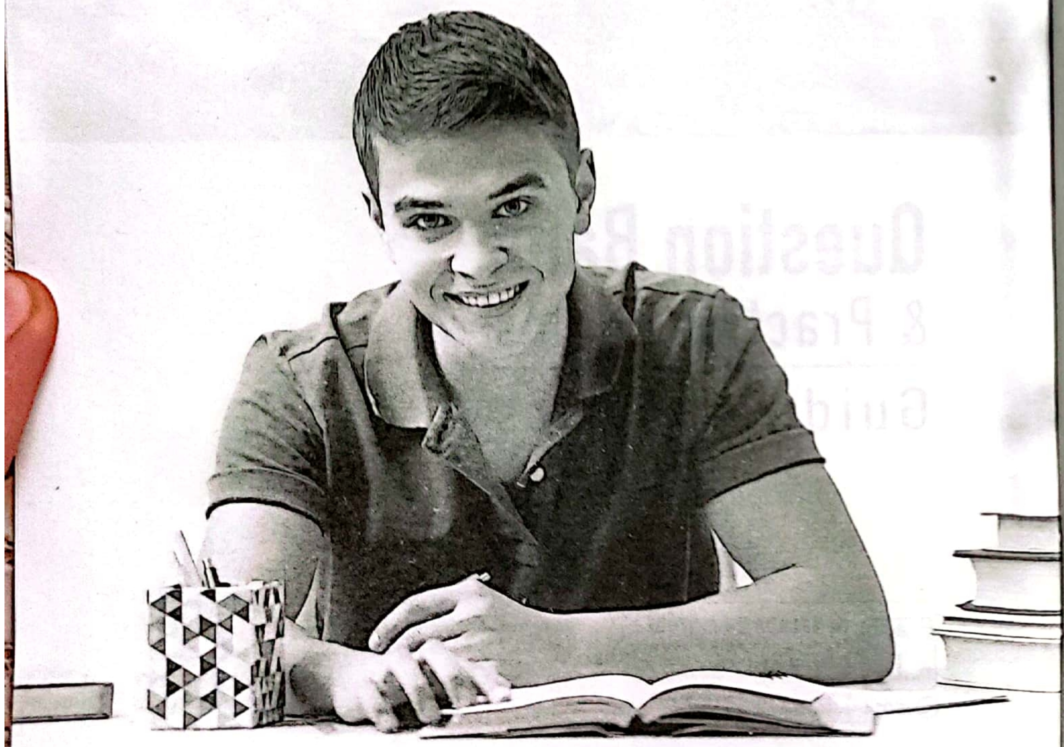
3rd Sec

Answers

of

Multiple Choice Question Bank



First: Questions on limits

1 (d)	2 (c)	3 (a)	4 (b)	5 (a)
6 (a)	7 (d)	8 (b)	9 (b)	10 (b)
11 (c)	12 (c)	13 (d)	14 (b)	15 (b)
16 (c)	17 (d)	18 (a)	19 (c)	20 (d)
21 (b)	22 (c)	23 (b)	24 (d)	25 (a)
26 (d)	27 (c)	28 (c)	29 (a)	30 (d)
31 (d)	32 (b)	33 (b)	34 (b)	35 (a)
36 (c)	37 (b)	38 (b)	39 (d)	40 (b)
41 (d)	42 (d)	43 (c)	44 (c)	45 (d)
46 (b)	47 (b)	48 (c)	49 (c)	50 (b)
51 (a)	52 (a)	53 (c)	54 (c)	55 (c)
56 (b)	57 (c)			

Second: Questions on derivatives - Implicit and parametric differentiation - Higher derivatives

1 (d)	2 (b)	3 (a)	4 (d)	5 (a)
6 (a)	7 (d)	8 (c)	9 (d)	10 (d)
11 (b)	12 (d)	13 (a)	14 (b)	15 (d)
16 (b)	17 (a)	18 (c)	19 (c)	20 (c)
21 (b)	22 (d)	23 (d)	24 (a)	25 (b)
26 (d)	27 (a)	28 (c)	29 (c)	30 (a)
31 (a)	32 (c)	33 (a)	34 (c)	35 (c)
36 (c)	37 (c)	38 (b)	39 (b)	40 (c)
41 (a)	42 (c)	43 (b)	44 (c)	45 (c)
46 (a)	47 (a)	48 (d)	49 (c)	50 (c)
51 (b)	52 (d)	53 (a)	54 (c)	55 (b)
56 (b)	57 (b)	58 (d)	59 (c)	60 (c)
61 (a)	62 (b)	63 (b)	64 (d)	65 (d)
66 (b)	67 (c)	68 (b)	69 (b)	70 (b)
71 (c)	72 (d)	73 (b)	74 (a)	75 (a)
76 (a)	77 (d)	78 (c)	79 (b)	80 (b)
81 (a)	82 (b)	83 (b)	84 (d)	85 (b)
86 (c)	87 (a)	88 (b)	89 (a)	90 (b)
91 (a)	92 (c)	93 (c)	94 (b)	95 (a)
96 (c)	97 (d)	98 (a)	99 (c)	100 (a)
101 (a)	102 (a)	103 (a)	104 (b)	105 (b)
106 (c)	107 (b)	108 (c)	109 (c)	110 (b)
111 (b)	112 (c)	113 (b)	114 (c)	115 (b)
116 (c)	117 (d)	118 (a)	119 (d)	120 (c)
121 (b)	122 (b)	123 (c)	124 (d)	125 (d)
126 (d)	127 (d)	128 (c)	129 (b)	130 (b)
131 (b)	132 (a)	133 (c)	134 (a)	135 (d)
136 (b)	137 (b)	138 (b)	139 (a)	140 (c)
141 (c)	142 (a)	143 (b)	144 (c)	145 (d)
146 (c)	147 (d)	148 (c)	149 (c)	150 (b)
151 (c)	152 (d)	153 (c)	154 (b)	155 (a)
156 (d)	157 (a)	158 (b)	159 (c)	160 (c)
161 (c)	162 (d)	163 (b)	164 (c)	165 (a)
166 (b)	167 (b)	168 (d)	169 (d)	170 (b)
171 (b)	172 (a)	173 (a)	174 (b)	175 (a)
176 (d)	177 (a)	178 (a)	179 (a)	180 (b)
181 (b)	182 (a)	183 (a)	184 (d)	185 (a)
186 (b)	187 (b)	188 (a)	189 (d)	190 (d)
191 (c)	192 (c)	193 (c)	194 (a)	195 (d)
196 (a)	197 (a)	198 (d)	199 (d)	200 (a)
201 (d)	202 (b)	203 (a)	204 (d)	205 (b)
206 (a)	207 (c)	208 (a)	209 (a)	210 (d)
211 (b)	212 (b)	213 (d)	214 (b)	215 (d)
216 (a)	217 (b)	218 (c)	219 (d)	220 (a)
221 (d)	222 (a)	223 (d)	224 (a)	225 (c)
226 (a)	227 (b)	228 (a)	229 (c)	230 (c)
231 (b)	232 (a)	233 (d)	234 (d)	235 (a)
236 (d)	237 (c)	238 (d)	239 (c)	240 (b)
241 (d)	242 (c)	243 (b)	244 (a)	245 (c)

246 First : (d) Second : (c) Third : (c)
Fourth : (c) Fifth : (a) Sixth : (b)

Third: Questions on the geometric applications (equation of tangent and normal)

1 (a)	2 (a)	3 (d)	4 (b)	5 (c)
6 (c)	7 (d)	8 (b)	9 (a)	10 (a)
11 (a)	12 (d)	13 (b)	14 (b)	15 (c)

16 ⓑ 17 ⓒ 18 ⓑ 19 ⓒ 20 ⓑ
21 ⓐ 22 ⓑ 23 ⓒ 24 ⓓ 25 ⓑ
26 ⓒ 27 ⓒ 28 ⓐ 29 ⓐ 30 ⓐ
31 ⓒ 32 ⓐ 33 ⓐ 34 ⓑ 35 ⓓ
36 ⓒ 37 ⓓ 38 ⓒ 39 ⓒ 40 ⓒ
41 ⓑ 42 ⓓ 43 ⓐ 44 ⓐ 45 ⓒ
46 ⓓ 47 ⓓ 48 ⓑ 49 ⓑ 50 ⓑ
51 ⓑ 52 ⓐ 53 ⓒ 54 ⓐ 55 ⓓ
56 ⓒ 57 ⓓ 58 ⓓ 59 ⓓ 60 ⓒ
61 ⓑ 62 ⓒ 63 ⓑ 64 ⓐ 65 ⓓ
66 ⓓ 67 ⓑ 68 ⓑ 69 ⓓ 70 ⓐ
71 ⓑ 72 ⓒ 73 ⓓ 74 ⓑ 75 ⓓ
76 ⓐ 77 ⓐ 78 ⓒ 79 ⓒ 80 ⓒ

Fourth — Questions on related time rates

1 ⓑ 2 ⓑ 3 ⓒ 4 ⓓ 5 ⓑ
6 ⓑ 7 ⓒ 8 ⓓ 9 ⓓ 10 ⓐ
11 ⓐ 12 ⓑ 13 ⓐ 14 ⓑ 15 ⓑ
16 ⓒ 17 ⓑ 18 ⓒ 19 ⓐ 20 ⓓ
21 ⓓ 22 ⓐ 23 ⓑ 24 ⓑ 25 ⓑ
26 ⓓ 27 ⓑ 28 ⓒ 29 ⓒ 30 ⓐ
31 ⓒ 32 First : ⓓ Second : ⓓ
Third : ⓑ Fourth : ⓓ
33 ⓐ 34 ⓒ 35 ⓐ 36 ⓒ 37 ⓓ
38 ⓒ 39 ⓓ 40 ⓑ 41 ⓒ 42 ⓓ
43 First : ⓐ Second : ⓑ 44 ⓐ
45 ⓓ 46 ⓒ 47 ⓑ 48 ⓓ 49 ⓑ
50 ⓐ 51 ⓒ 52 ⓐ 53 ⓐ 54 ⓒ
55 ⓑ 56 ⓓ 57 ⓓ 58 ⓒ 59 ⓒ
60 ⓒ 61 ⓑ 62 ⓐ 63 ⓐ 64 ⓑ
65 ⓑ 66 ⓐ 67 ⓐ 68 ⓐ 69 ⓑ
70 ⓐ 71 ⓓ 72 ⓐ 73 ⓒ 74 ⓑ
75 ⓒ 76 ⓑ 77 ⓒ 78 ⓒ 79 ⓐ
80 ⓒ 81 ⓑ 82 ⓐ 83 ⓑ 84 ⓓ
85 First : ⓑ Second : ⓒ
Third : ⓐ Fourth : ⓑ
86 ⓒ 87 ⓒ 88 ⓑ

Fifth — Questions of behavior of the function

1

1 ⓑ 2 ⓑ 3 ⓐ 4 ⓑ 5 ⓒ
6 ⓑ 7 ⓐ 8 ⓒ 9 ⓒ 10 ⓑ
11 ⓑ 12 ⓓ 13 ⓒ 14 ⓑ 15 ⓓ
16 ⓐ 17 ⓐ 18 ⓐ 19 ⓓ 20 ⓐ
21 ⓓ 22 ⓓ 23 ⓓ 24 ⓒ 25 ⓑ
26 ⓑ 27 ⓓ 28 ⓒ 29 ⓒ 30 ⓓ
31 ⓐ 32 ⓒ 33 ⓐ 34 ⓑ 35 ⓒ
36 ⓐ 37 ⓐ 38 ⓑ 39 ⓒ 40 ⓓ
41 ⓐ 42 ⓓ 43 ⓑ 44 ⓓ 45 ⓑ
46 ⓓ 47 ⓐ 48 ⓑ 49 ⓐ 50 ⓐ
51 ⓒ 52 ⓓ 53 ⓑ 54 ⓑ 55 ⓓ
56 ⓑ 57 ⓓ 58 ⓓ 59 ⓓ 60 ⓑ
61 ⓑ 62 ⓐ 63 ⓐ 64 ⓓ 65 ⓐ
66 ⓒ 67 ⓐ 68 ⓐ 69 ⓐ 70 ⓐ
71 ⓑ 72 ⓐ 73 ⓒ 74 ⓐ 75 ⓓ
76 ⓒ 77 ⓑ 78 ⓑ 79 ⓑ 80 ⓐ
81 ⓒ 82 ⓐ 83 ⓓ 84 ⓐ 85 ⓑ
86 ⓓ 87 ⓒ 88 ⓑ 89 ⓒ 90 ⓑ
91 ⓓ 92 ⓒ 93 ⓓ 94 ⓓ 95 ⓒ
96 ⓑ 97 ⓓ 98 ⓑ 99 ⓒ 100 ⓐ
101 ⓑ 102 ⓒ 103 ⓒ 104 ⓒ 105 ⓑ
106 ⓐ 107 ⓑ

2

1 ⓑ 2 ⓒ 3 ⓑ
4 ⓑ 5 ⓒ 6 ⓑ

Sixth — Questions on applications of maxima and minima

1 ⓒ 2 ⓐ 3 ⓑ 4 ⓒ 5 ⓒ
6 ⓐ 7 ⓑ 8 ⓑ 9 ⓒ 10 ⓐ
11 ⓓ 12 ⓑ 13 ⓑ 14 ⓐ 15 ⓓ
16 ⓑ 17 ⓒ 18 ⓑ 19 ⓒ 20 ⓓ
21 ⓐ 22 ⓑ 23 ⓒ 24 ⓒ 25 ⓑ
26 ⓒ 27 ⓓ 28 ⓑ 29 ⓑ 30 ⓑ

31 ⓒ 32 ⓑ 33 ⓐ 34 ⓐ 35 ⓐ
36 ⓑ 37 ⓑ 38 ⓐ 39 ⓒ 40 ⓓ
41 ⓑ 42 ⓐ 43 ⓓ 44 ⓒ 45 ⓒ
46 ⓑ 47 ⓒ 48 ⓐ 49 ⓑ 50 ⓐ
51 ⓒ 52 ⓑ 53 ⓒ 54 ⓑ 55 ⓓ
56 ⓒ 57 ⓒ 58 ⓐ 59 ⓐ 60 ⓓ
61 ⓑ 62 ⓑ 63 ⓒ 64 ⓐ 65 ⓒ
66 ⓑ 67 ⓑ 68 ⓑ 69 ⓑ 70 ⓐ
71 ⓑ 72 ⓒ 73 ⓒ 74 ⓑ 75 ⓐ
76 ⓑ 77 ⓐ

Seventh — Questions on behavior of the curves represent graphically

1 First : ⓒ Second : ⓑ
Third : ⓐ Fourth : ⓑ
Fifth : ⓑ
2 ⓐ 3 ⓐ 4 ⓓ 5 ⓓ 6 ⓓ
7 ⓐ 8 ⓒ 9 ⓑ
10 First : ⓑ Second : ⓓ
Third : ⓒ Fourth : ⓐ
Fifth : ⓒ Sixth : ⓑ
11 ⓒ
12 First : ⓒ Second : ⓑ
13 First : ⓓ Second : ⓐ
14 ⓒ 15 ⓑ 16 ⓐ 17 ⓓ 18 ⓑ
19 ⓑ
20 First : ⓒ Second : ⓓ
21 ⓑ
22 First : ⓓ Second : ⓓ
Third : ⓓ
23 First : ⓑ Second : ⓑ
Third : ⓓ Fourth : ⓓ
24 ⓒ 25 ⓓ 26 ⓓ 27 ⓓ 28 ⓒ
29 ⓓ 30 ⓒ 31 ⓐ 32 ⓒ 33 ⓐ
34 ⓐ 35 ⓐ 36 ⓐ 37 ⓐ 38 ⓐ
39 ⓒ 40 ⓓ 41 ⓑ 42 ⓐ 43 ⓒ
44 ⓓ

Eighth — Questions on the integration

1 ⓑ 2 ⓑ 3 ⓐ 4 ⓑ 5 ⓑ
6 ⓓ 7 ⓑ 8 ⓓ 9 ⓑ 10 ⓓ
11 ⓑ 12 ⓓ 13 ⓓ 14 ⓒ 15 ⓒ
16 ⓐ 17 ⓑ 18 ⓓ 19 ⓐ 20 ⓑ
21 ⓓ 22 ⓑ 23 ⓓ 24 ⓐ 25 ⓒ
26 ⓒ 27 ⓒ 28 ⓑ 29 ⓑ 30 ⓐ
31 ⓒ 32 ⓐ 33 ⓑ 34 ⓑ 35 ⓑ
36 ⓑ 37 ⓑ 38 ⓐ 39 ⓑ 40 ⓓ
41 ⓑ 42 ⓐ 43 ⓑ 44 ⓒ 45 ⓑ
46 ⓐ 47 ⓓ 48 ⓒ 49 ⓓ 50 ⓐ
51 ⓑ 52 ⓒ 53 ⓑ 54 ⓐ 55 ⓒ
56 ⓓ 57 ⓑ 58 ⓒ 59 ⓓ 60 ⓒ
61 ⓑ 62 ⓓ 63 ⓑ 64 ⓒ 65 ⓐ
66 ⓑ 67 ⓓ 68 ⓑ 69 ⓐ 70 ⓓ
71 ⓒ 72 ⓐ 73 ⓑ 74 ⓓ 75 ⓑ
76 ⓒ 77 ⓐ 78 ⓓ 79 ⓑ 80 ⓐ
81 ⓓ 82 ⓐ 83 ⓒ 84 ⓒ 85 ⓒ
86 ⓓ 87 ⓒ 88 ⓒ 89 ⓐ 90 ⓐ
91 ⓐ 92 ⓑ 93 ⓐ 94 ⓑ 95 ⓑ
96 ⓓ 97 ⓒ 98 ⓓ 99 ⓐ 100 ⓓ
101 ⓑ 102 ⓐ 103 ⓑ 104 ⓑ 105 ⓓ
106 ⓐ 107 ⓓ 108 ⓒ 109 ⓓ 110 ⓑ
111 ⓐ 112 ⓐ 113 ⓓ 114 ⓑ 115 ⓓ
116 ⓒ 117 ⓑ 118 ⓒ 119 ⓓ 120 ⓐ
121 ⓐ 122 ⓑ 123 ⓓ 124 ⓑ 125 ⓒ
126 ⓑ 127 ⓒ 128 ⓑ 129 ⓓ 130 ⓑ
131 ⓐ 132 ⓐ 133 ⓒ 134 ⓐ 135 ⓐ
136 ⓑ 137 ⓑ 138 ⓒ 139 ⓐ 140 ⓐ
141 ⓓ 142 ⓐ 143 ⓓ 144 ⓐ 145 ⓓ
146 ⓐ 147 ⓑ 148 ⓐ 149 ⓑ 150 ⓒ
151 ⓒ

Ninth — Questions on the definite integration

1 ⓒ 2 ⓑ 3 ⓑ 4 ⓒ 5 ⓐ
6 ⓐ 7 ⓑ 8 ⓐ 9 ⓒ 10 ⓒ
11 ⓓ 12 ⓑ 13 ⓑ 14 ⓑ 15 ⓑ

16 b	17 b	18 b	19 d	20 d
21 c	22 c	23 b	24 b	25 b
26 b	27 a	28 b	29 a	30 c
31 b	32 a	33 d	34 b	35 c
36 b	37 c	38 c	39 c	40 c
41 c	42 a	43 c	44 a	45 c
46 d	47 c	48 a	49 d	50 a
51 c	52 b	53 b	54 c	55 b
56 c	57 d	58 d	59 c	60 a
61 c	62 d	63 a	64 b	65 b
66 b	67 c	68 d	69 c	70 d
71 b	72 b	73 c	74 b	75 b
76 b	77 b	78 b	79 d	80 a
81 b	82 d	83 c	84 a	85 b
86 c	87 c	88 b	89 d	

Tenth Questions on the applications of integration

1 a	2 d	3 a	4 d	5 d
6 b	7 c	8 d	9 b	10 d
11 a	12 b	13 a	14 c	15 d
16 a	17 a	18 d	19 d	20 d
21 c	22 b	23 c	24 a	25 c
26 b	27 a	28 c	29 b	30 b
31 b	32 c	33 c	34 d	35 b
36 b	37 b	38 b	39 c	40 c
41 a	42 b	43 d	44 b	45 b
46 b				

Eleventh Questions on the areas and the volumes of revolution solids

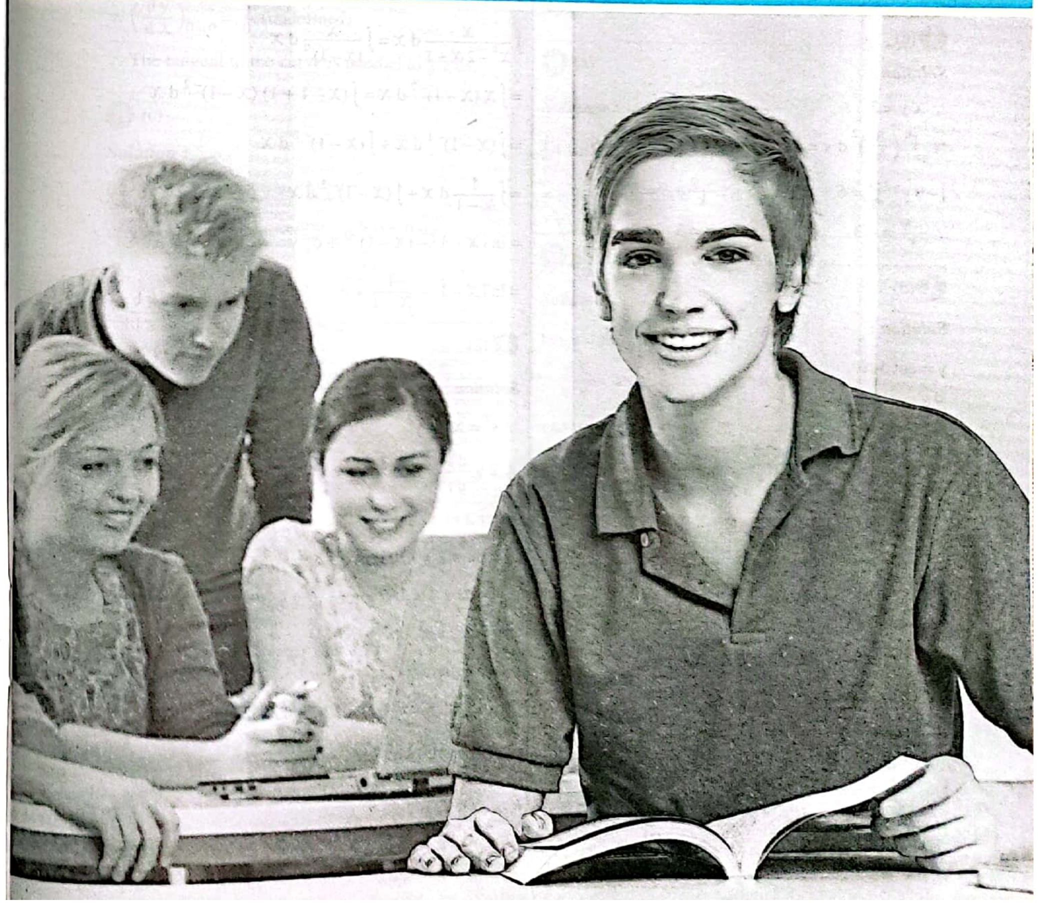
1 a	2 d	3 b	4 a	5 b
6 c	7 a	8 a	9 b	10 a
11 a	12 b	13 c	14 b	15 c
16 c	17 c	18 c	19 a	20 c
21 b	22 c	23 a	24 d	25 b
26 b	27 b	28 a	29 c	30 b
31 c	32 d	33 c	34 d	35 b
36 c	37 b	38 c	39 c	40 d
41 b	42 d	43 a	44 c	45 b
46 c	47 d	48 d	49 b	50 c
51 b	52 d	53 c	54 d	55 c
56 c	57 d	58 c	59 c	60 d
61 b	62 d	63 b	64 a	65 c
66 d	67 b	68 b	69 b	70 a
71 c	72 c	73 b	74 a	75 d
76 a	77 a	78 c	79 c	80 c
81 c	82 d	83 a	84 c	85 b
86 a	87 a	88 b	89 c	90 d
91 c	92 c	93 a	94 d	95 a
96 c	97 a	98 b	99 b	100 c
101 a	102 d	103 c		

104 First : d   Second : c

Third : d

105 a	106 d	107 d	108 c	109 d
110 b				

Answers of Practice Exams



Answers of Practice Exams

Exam 1

1 (d)

Solution :

$\lim\limits_{x\to 0} \sqrt[x]{1+x} = \lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$

2 (d)

Solution :

Circumference of the circle (P) $= 2\pi r$

$\therefore \frac{dp}{dt} = 2\pi \frac{dr}{dt} = 2\pi \times \frac{4}{\pi} = 8$ cm./sec.

3 (c)

Solution :

$\because xy = 3 \qquad \therefore x = \frac{3}{y}$

$\pi \int_1^k \left(\frac{3}{y}\right)^2 dy = 6\pi \qquad \therefore \int_1^k 9y^{-2}\, dy = 6$

$[-9y^{-1}]_1^k = 6 \qquad \therefore \frac{-9}{k} + 9 = 6$

$\therefore \frac{-9}{k} = -3 \qquad \therefore k = 3$

4 (d)

Solution :

$y = \cot 5x$

$\frac{dy}{dx} = -5\csc^2 5x$

$\therefore \frac{dy}{dx} + 5y^2 = -5\csc^2 5x + 5\cot^2 5x$

$= -5(\csc^2 5x - \cot^2 5x)$

$= -5 \times 1 = -5$

5 (a)

Solution :

$\because f(x) = x^3 - 3x - 4$ is decreasing

$, f'(x) = 3x^2 - 3 \qquad \therefore f'(x) < 0$

$\therefore 3x^2 - 3 < 0 \qquad \therefore 3x^2 < 3$

$\therefore x^2 < 1 \qquad \therefore |x| < 1$

6 (b)

Solution :

$y^2 = 1 - x^{-2}$ (by differentiation with respect to x)

$\therefore 2y\frac{dy}{dx} = 2x^{-3}$

$\therefore y\frac{dy}{dx} = x^{-3}$ (by differentiation with respect to x)

$\therefore y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -3x^{-4}$

7 (a)

Solution :

$\int \frac{x}{x^2 - 2x + 1}\, dx = \int \frac{x}{(x-1)^2}\, dx$

$= \int x(x-1)^{-2}\, dx = \int (x - 1 + 1)(x-1)^{-2}\, dx$

$= \int (x-1)^{-1}\, dx + \int (x-1)^{-2}\, dx$

$= \int \frac{1}{x-1}\, dx + \int (x-1)^{-2}\, dx$

$= \ln|x-1| - (x-1)^{-1} + c$

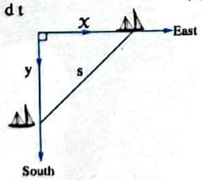
$= \ln|x-1| - \frac{1}{x-1} + c$

8 (a)

Solution :

$\because s^2 = x^2 + y^2$

$\therefore 2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$ (1)



After 2 hr. :

$x = 60 \times 2 = 120$ km.

$, y = 80 \times 2 = 160$ km.

$, s = 200$ km.

From equation (1) :

$\therefore 2 \times 200 \times \frac{ds}{dt} = 2 \times 120 \times 60 + 2 \times 160 \times 80$

$\therefore \frac{ds}{dt} = 100$ km./hr.

9 (d)

Solution :

The shaded area $= \int_{-1}^{4} [g(x) - f(x)]\, dx$

$= \int_{-1}^{4} g(x)\, dx - \int_{-1}^{4} f(x)\, dx$

$= 150 - [(x-1)^3]_{-1}^{4}$

$= 150 - (27 - (-8)) = 115$ square unit

10 (a)

Solution :

$\because \frac{dx}{d\theta} = -2\sin 2\theta , \frac{dy}{d\theta} = 3\cos 3\theta$

$\therefore \frac{dy}{dx} = \frac{3\cos 3\theta}{-2\sin 2\theta}$

$\therefore \left(\frac{dy}{dx}\right)_{\theta=0} = \frac{3}{0}$ (undefined)

$\therefore$ The tangent to the curve is parallel to y-axis.

11 (c)

Solution :

$y = e^x \times e^{2x} \times e^{3x} \times \dots \times e^{10x}$

$\therefore y = e^{(x + 2x + 3x + \dots + 10x)} = e^{\frac{10}{2}(x + 10x)}$

$= e^{55x}$

$\therefore \frac{dy}{dx} = 55e^{55x}$

$\therefore \frac{d^2y}{dx^2} = (55)^2 e^{55x} = a e^{bx}$

$\therefore \frac{a}{b} = \frac{(55)^2}{55} = 55$

12 (d)

Solution :

$\int_a^{\frac{\pi}{2}} \frac{\sin x(1 + \cos 2x)}{\sin 2x}\, dx$

$= \int_a^{\frac{\pi}{2}} \frac{\sin x(1 + 2\cos^2 x - 1)}{2\sin x\cos x}\, dx$

$= \int_a^{\frac{\pi}{2}} \cos x\, dx = [\sin x]_a^{\frac{\pi}{2}}$

$= 1 - \sin a$

$\therefore 1 - \sin a = \frac{1}{2} \qquad \therefore \sin a = \frac{1}{2}$

$\therefore a = \frac{\pi}{6}$

13 (d)

Solution :

Put $x = 0$ in the equation of the curve

$\therefore y = b$

$\therefore$ Point of intersection with y-axis is $(0 , b)$

, slope of tangent $= \frac{dy}{dx} = \frac{-b}{a} e^{-\frac{x}{a}}$

$\left(\frac{dy}{dx}\right)_{x=0} = \frac{-b}{a}$

$\therefore$ Equation of tangent : $\frac{y-b}{x} = \frac{-b}{a}$

$\therefore ay - ab = -bx$

$\therefore bx + ay = ab$ (divide by a b)

$\therefore \frac{x}{a} + \frac{y}{b} = 1$

14 (a)

Solution :

$\int 4\sin 2x\, e^{\cos 2x}\, dx$

$= -2\int -2\sin 2x\, e^{\cos 2x}\, dx = -2e^{\cos 2x} + c$

15 (c)

Solution :

Let $y = \sin^2 x$

$\therefore \frac{dy}{dx} = 2\sin x\cos x$

$, z = \cos^2 x$

$\therefore \frac{dz}{dx} = -2\sin x\cos x$

$\therefore \frac{dy}{dz} = \frac{dy}{dx} \div \frac{dz}{dx} = \frac{2\sin x\cos x}{-2\sin x\cos x}$

$= -1 = -(\sin^2 x + \cos^2 x)$

Another solution : Let $y = \sin^2 x , z = \cos^2 x$

$\therefore y + z = 1 \qquad \therefore \frac{dy}{dz} + 1 = 0$

$\therefore \frac{dy}{dz} = -1$

$\therefore \frac{dy}{dz} = -(\sin^2 x + \cos^2 x)$

16 (c)

Solution :

$\int (1+\cos x)^2\, dx$

$= \int (1+2\cos x+\cos^2 x)\, dx$

$= \int \left(1+2\cos x+\frac{1}{2}+\frac{1}{2}\cos 2x\right) dx$

$= \int \left(\frac{3}{2}+2\cos x+\frac{1}{2}\cos 2x\right) dx$

$= \frac{3}{2}x+2\sin x+\frac{1}{4}\sin 2x+c$

17 (c)

Solution :

$\int \frac{e^{ax}}{e^{bx}}\, dx = \int e^{(a-b)x}\, dx$

$= \frac{1}{a-b} e^{(a-b)x}+c$

$\therefore k = \frac{1}{a-b}$

18 (b)

Solution :

$f(x) = 3ax^3 - bx - 5$

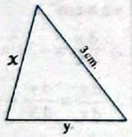
$\therefore f'(x) = 9ax^2 - b$, at $x = 1$ has local maximum value

$\therefore f'(1) = 0$

$\therefore 9a(1) - b = 0 \qquad \therefore \frac{b}{a} = 9$

19 (a)

Solution :



Let the length of unknown sides of the triangle are x , y

$\therefore x+y+3 = 8$

$\therefore y = 5-x$

Area of triangle (A) $= \sqrt{4(4-3)(4-x)(4-y)}$

$= \sqrt{4(4-x)(4-5+x)}$

$= \sqrt{4(4-x)(x-1)}$ (1)

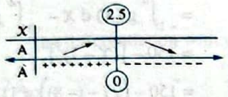
$\therefore A = \sqrt{-4x^2+20x-16}$

$\therefore \frac{dA}{dx} = \frac{-8x+20}{2\sqrt{-4x^2+20x-16}}$

put $\frac{dA}{dx} = 0$

$\therefore x = 2.5$

From (1) : $A = \sqrt{4(4-2.5)(2.5-1)} = 3$



$\therefore$ Maximum area of the triangle $= 3\ cm^2$.

20 (d)

Solution :

$f(x) = \begin{cases} x^2+x-2 & , \quad x \le 2 \\ 5x-6 & , \quad x > 2 \end{cases}$

Differentiate by using the definition

$f'(2^+) = f'(2^-) = 5$

$\therefore f'(x) = \begin{cases} 2x+1 & , \quad x < 2 \\ 5 & , \quad x = 2 \\ 5 & , \quad x > 2 \end{cases}$

, put $f'(x) = 0$, when $x < 2$

$\therefore 2x+1 = 0 \qquad \therefore x = \frac{-1}{2} \in [-1 , 3]$

, in case $x \ge 2$, then : $f'(x) \ne$ zero

, $\because f(-1) = -2$, $f\left(\frac{-1}{2}\right) = \frac{-9}{4}$, $f(3) = 9$

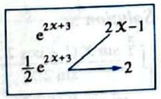
$\therefore$ The function has absolute maximum value $= 9$ at $x = 3$

and it has absolute minimum value $= \frac{-9}{4}$ at $x = \frac{-1}{2}$

$\therefore a+b = 9 - \frac{9}{4} = 6.75$

21 (b)

Solution :



$\int (2x-1)e^{2x+3}\, dx$

$= (2x-1)\left(\frac{1}{2}e^{2x+3}\right) - \int e^{2x+3}\, dx$

$\therefore \int z\, dy = \int e^{2x+3}\, dx$

$= \frac{1}{2}e^{2x+3}+c$

22 (a)

23 (b)

Solution :

$f'(x) = \frac{a}{x}+2bx+1$

, $\because$ at $x = -1$, $x = 2$ critical points

$\therefore f'(-1) = -a-2b+1 = 0$

$\therefore a+2b = 1$ (1)

, $f'(2) = \frac{1}{2}a+4b+1 = 0$

$\therefore \frac{1}{2}a+4b = -1$ (2)

From (1) , (2) : $a = 2$, $b = -\frac{1}{2}$

$\therefore ab = -1$

24 (c)

Solution :

$\because \frac{d^2x}{dy^2} = e^{y+1}$ (integrate with respect to y)

$\therefore \frac{dx}{dy} = \int e^{y+1}\, dy = e^{y+1}+c$

$\because$ The tangent is parallel to the line $y = x-3$

$\therefore$ Its slope $= 1$

$\therefore \frac{dy}{dx} = 1$ then $\left(\frac{dx}{dy}\right)_{(x , -1)} = 1$

$\therefore 1 = e^{-1+1}+c \qquad \therefore c = 0 \qquad \therefore \frac{dx}{dy} = e^{y+1}$

$\therefore$ The slope of the tangent at any point is $\frac{dy}{dx} = e^{-y-1}$

25 (d)

Solution :

The curve is convex upwards for all the values of $x \in \mathbb{R} - \{3\}$

In figure (D)

Exam 2

1 (a)

Solution :

$\lim_{x \to 0} \frac{e^{5x}-1}{x} = 5 \lim_{5x \to 0} \frac{e^{5x}-1}{5x} = 5 \times 1 = 5$

2 (c)

Solution :

$f(x) = \sin 2x$

$\therefore f'(x) = 2\cos 2x$

, $f''(x) = -4\sin 2x$

$\therefore f''\left(\frac{\pi}{4}\right) = -4$

3 (b)

Solution :

$\int \sec x\, dx = \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\, dx$

$= \int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\, dx$

$= \ln|\sec x+\tan x|+c$

4 (b)

Solution :

Slope of the tangent $= \frac{dy}{dx} = x\sqrt{x^2+1}$

$\therefore y = \int x(x^2+1)^{\frac{1}{2}}\, dx = \frac{1}{2}\int (2x)(x^2+1)^{\frac{1}{2}}\, dx$

$= \frac{1}{2} \times \frac{2}{3}(x^2+1)^{\frac{3}{2}}+c = \frac{1}{3}(x^2+1)^{\frac{3}{2}}+c$

$\because$ The point $(0 , 1) \in$ the curve

$\therefore 1 = \frac{1}{3}(0+1)^{\frac{3}{2}}+c \qquad \therefore c = \frac{2}{3}$

$\therefore$ The equation of the curve : $y = \frac{1}{3}(x^2+1)^{\frac{3}{2}}+\frac{2}{3}$

5 (a)

Solution :

$\int_0^4 y\, dx = 68 \qquad \therefore \int_0^4 (x^3+a)\, dx = 68$

$\therefore \left[\frac{1}{4}x^4+ax\right]_0^4 = 68 \qquad \therefore 64+4a = 68$

$\therefore 4a = 4 \qquad \therefore a = 1$

6 (b)

Solution :

Let the side length of the square ℓ

$\therefore$ The area of the shaded part $= \frac{1}{2}$ the area of the square $= \frac{1}{2}\ell^2$

$\therefore \frac{dA}{dt} = \ell\frac{d\ell}{dt} = 16 \times \frac{1}{4} = 4\ cm^2/sec.$

7 (c)

Solution :

$\sin x \cos y - \sin y \cos x = 1$ $\quad \therefore \sin(x-y) = 1$

$\therefore x - y = \frac{\pi}{2}$

Differentiate with respect to x :

$\therefore 1 - \frac{dy}{dx} = 0$ $\quad \therefore \frac{dy}{dx} = 1$

---

8 (d)

Solution :

$f(x) = x^3 + a x^2 + 12 x + 1$

$\therefore f'(x) = 3x^2 + 2 a x + 12$

$\because$ The function has no critical points

$\therefore 3x^2 + 2 a x + 12 \neq 0$

$\therefore$ Discriminant < zero

$\therefore (2a)^2 - 4 \times 3 \times 12 < 0$

$\therefore 4 a^2 < 144$ $\quad \therefore a^2 < 36$

$\therefore |a| < 6$ $\quad \therefore a \in]-6, 6[$

---

9 (d)

Solution :

The volume from rotation

$= \pi \int_{y=0}^{y=a} x^2 \, dy = \pi \int_0^a (a - y) \, dy$

$= \pi \left[a y - \frac{1}{2} y^2\right]_0^a = \pi \left(\left(\frac{1}{2} a^2\right) - (0)\right) = \frac{1}{2} a^2 \pi$

$\therefore \frac{1}{2} a^2 \pi = 8 \pi$ $\quad \therefore a^2 = 16$ $\quad \therefore a = 4$

---

10 (c)

Solution :

$\because f(x) = \sqrt{x^2 + 9}$

$\therefore f'(x) = \frac{2x}{2\sqrt{x^2+9}}$

$= \frac{x}{\sqrt{x^2+9}}$

, putting $f'(x) = 0$ $\quad \therefore x = 0$

, $\because f(0) = 3$, $f(-3) = f(3) = 3\sqrt{2}$

$\therefore$ The absolute minimum value $= 3$

---

11 (c)

Solution :

$\because x y = x^y$ $\quad \therefore \ln(x y) = y \ln x$

$\therefore \ln x + \ln y = y \ln x$ "differentiate with respect to x"

$\therefore \frac{1}{x} + \frac{y'}{y} = \frac{y}{x} + y' \ln x$

at $(1, 1)$ $\quad \therefore 1 + y' = 1 + 0$

$\therefore y' = 0$

$\therefore$ The equation of tangent is $\frac{y-1}{x-1} = 0$

$\therefore y - 1 = 0$

---

12 (b)

Solution :

$f\left(\frac{1}{2} x\right) = |x|^3$ $\quad \therefore f(x) = |2x|^3$

$\therefore f'(x) = 6|2x|^2$

$\therefore f''(x) = 24|2x|$

$\therefore f''(-1) = 48$

---

13 (b)

Solution :

$y = x^{n+1} + n x^{n-1} + 1$

$\therefore \frac{dy}{dx} = (n+1) x^n + n(n-1) x^{n-2}$

$\therefore \frac{d^2y}{dx^2} = (n+1)(n) x^{n-1} + n(n-1)(n-2) x^{n-3}$

, $\frac{d^ny}{dx^n} = (n+1)(n)(n-1) \ldots\ldots 3 \times 2 \times x + \text{zero}$

$= \lfloor n+1 \, x$

---

14 (b)

Solution :

$\because z = x + 2$

$\therefore x = z - 2$, $dx = dz$

$\therefore \int \frac{x-2}{x^2+4x+4} \, dx = \int \frac{z-4}{z^2} \, dz$

$= \int \left(\frac{1}{z} - 4 z^{-2}\right) dz = \ln|z| + 4 z^{-1} + c$

$= \ln|z| + \frac{4}{z} + c$

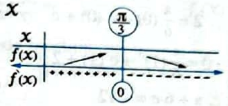
15 (d)

Solution :

$\because f(x) = \sin x + \frac{1}{2} \sin 2x$

$\therefore f'(x) = \cos x + \cos 2x$

, putting $f'(x) = 0$



$\therefore \cos 2x + \cos x = 0$

$\therefore 2\cos^2 x + \cos x - 1 = 0$

$\therefore (2\cos x - 1)(\cos x + 1) = 0$

$\therefore \cos x = \frac{1}{2}$, then $x = \frac{\pi}{3}$

or $\cos x = -1$, then $x = \pi$ (refused)

$\therefore$ At $x = \frac{\pi}{3}$ the function has local maximum value.

---

16 (a)

Solution :

$y = \frac{\log_x 2e}{\log_x 3e} = \frac{\log 2e}{\log x} \times \frac{\log x}{\log 3e}$

$= \frac{\log 2e}{\log 3e}$

$\therefore \frac{dy}{dx} = 0$

---

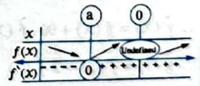
17 (c)

Solution :

$f(x) = x^2 + \frac{2a^3}{x}$

$\therefore f'(x) = 2x - \frac{2a^3}{x^2} = \frac{2x^3 - 2a^3}{x^2}$

put $f'(x) = 0$



$\therefore 2x^3 - 2a^3 = 0$

$\therefore x = a$

$\therefore f(x)$ is decreasing on $]-\infty, a[$

---

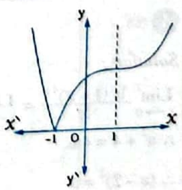
18 (a)

Solution :

$\because f'(x) = |g(x)|$

$\therefore$ The opposite figure

represents the curve $f'(x)$

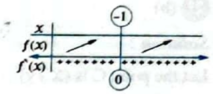


at $x = -1$ $\quad f'(x) = 0$

but it is not local

minimum value

$\therefore$ at $x = -1$ only

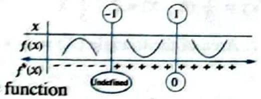


the function f

has inflection point

and the curve of the function

is not convex upward on $]-1, 1[$



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19 (c)

Solution :

Let $f(x) = \frac{\tan x}{x^2 + \cos x}$

$\therefore f(-x) = \frac{\tan(-x)}{(-x)^2 + \cos(-x)} = \frac{-\tan x}{x^2 + \cos x} = -f(x)$

$\therefore$ The function is odd

$\therefore \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(x) \, dx = \text{zero}$

---

20 (d)

Solution :

Equation of straight line passes through two points

$(3, -3)$, $(-2, 2)$ is $y = -x$

to find points of intersection of the curve y_1 and the straight line y_2 where

$y_1 = 6 - x^2$, $y_2 = -x$

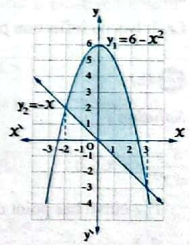
$\therefore 6 - x^2 = -x$

$\therefore x^2 - x - 6 = 0$

$\therefore (x - 3)(x + 2) = 0$

$\therefore x = 3$

or $x = -2$



$\therefore$ The area $= \int_{-2}^{3} (y_1 - y_2) \, dx$

where $y_1 > y_2$ in the interval $[-2, 3]$

$= \int_{-2}^{3} \left((6 - x^2) - (-x)\right) dx = \int_{-2}^{3} (-x^2 + x + 6) \, dx$

$= \left[-\frac{1}{3} x^3 + \frac{1}{2} x^2 + 6x\right]_{-2}^{3}$

$= \left[-\frac{1}{3}(3)^3 + \frac{1}{2}(3)^2 + 6(3)\right]$

$- \left[-\frac{1}{3}(-2)^3 + \frac{1}{2}(-2)^2 + 6(-2)\right] = \frac{125}{6}$ square units.

21 (b)

Solution :

Let the point C is (x , y)

, $y = \frac{1}{3}(6 - x) = 2 - \frac{1}{3}x$

$\therefore$ Area of the triangle (A) $= \frac{1}{2} \times 2x \times y$

$= 2x - \frac{1}{3}x^2$

$\therefore \frac{dA}{dx} = 2 - \frac{2}{3}x$

Put $\frac{dA}{dx} = 0$ $\qquad \therefore x = 3$

, $\frac{d^2A}{dx^2} = \frac{-2}{3} < 0$

$\therefore$ The Maximum area $= 2 \times 3 - \frac{1}{3}(3)^2 = 3$ square unit.

---

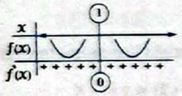
22 (c)

Solution :

$f(x) = (x-1)^4 + 3$ $\qquad \therefore f'(x) = 4(x-1)^3$

, $f''(x) = 12(x-1)^2$

put $f''(x) = 0$ $\qquad \therefore x = 1$



$\therefore$ The curve is convex downward in $]-\infty , 1[$ and $]1 , \infty[$

i.e. It has no inflection point.

---

23 (b)

Solution :

$\frac{d^2y}{dx^2} = ax + b$

$\because$ The curve has point of inflection at $x =$ zero

$\therefore \frac{d^2y}{dx^2} =$ zero $\qquad \therefore$ zero $=$ zero $+ b$

$\therefore b = 0$

$\therefore \frac{dy}{dx} = \int ax\,dx = \frac{a}{2}x^2 + c$

$\because f$ has minimum value at $(1 , 0)$

$\therefore \frac{dy}{dx} =$ zero , at $x = 1$

$\therefore 0 = \frac{a}{2} + c$, then $a = -2c$ $\qquad$ (1)

$\therefore y = \int (\frac{a}{2}x^2 + c)\,dx$

$= \frac{a}{6}x^3 + cx + \grave{c}$

$\because$ The curve passes through the points $(0 , 2)$, $(1 , 0)$

$\therefore 2 = \frac{a}{6}(0) + c(0) + \grave{c}$ $\qquad \therefore \grave{c} = 2$

, $0 = \frac{a}{6}(1)^3 + c(1) + 2$

$\therefore a + 6c = -12$ $\qquad$ (2)

By substitute from (1) in (2) :

$\therefore (-2c) + 6c = -12$ $\qquad \therefore 4c = -12$

$\therefore c = -3$ $\qquad a = 6$

$\therefore 2a + b = 12$

---

24 (c)

Solution :

$\int_1^a \frac{1}{x}\,dx = \int_b^1 \frac{1}{x}\,dx$

$\therefore [\ln|x|]_2^a = [\ln|x|]_b^1$, $\because a > b > 0$

$\therefore [\ln x]_1^a = [\ln x]_b^1$

$\therefore \ln a - \ln 1 = \ln 1 - \ln b$

$\therefore \ln a = -\ln b = \ln \frac{1}{b}$ $\qquad \therefore a = \frac{1}{b}$

---

25 (d)

Solution :

$f'(3) = \tan 45° = 1$

, $g(x) = x . f(x)$

$\therefore g'(x) = f(x) + x . f'(x)$

$\therefore g'(3) = f(3) + 3 f'(3)$

$= 5 + 3 \times 1 = 8$

Exam 3

1 (c)

Solution :

$\lim\limits_{x \to 0} \frac{\ln(1+x)^{4a}}{x} = \lim\limits_{x \to 0} \frac{4a \ln(1+x)}{x} = 4a \times 1$

$\therefore a^2 + 4 = 4a$ $\qquad \therefore a^2 - 4a + 4 = 0$

$\therefore (a-2)^2 = 0$ $\qquad \therefore a = 2$

2 (b)

Solution :

let $f(x) = \frac{4x + \sin x}{x^2 + \cos x}$

$\therefore f(-x) = \frac{4(-x) + \sin(-x)}{(-x)^2 + \cos(-x)} = \frac{-4x - \sin x}{x^2 + \cos x} = -f(x)$

$\therefore$ The function f is an odd function

$\therefore \int_{-\pi}^{\pi} f(x)\,dx =$ zero

---

3 (b)

Solution :

$f(x) = x^2 + ax + b$

$\therefore f'(x) = 2x + a$

$\because$ The function has local minimum value is 3 at $x = 1$

$\therefore f'(1) = 0$ $\qquad \therefore 2 + a = 0$

$\therefore a = -2$ $\qquad$, $f(1) = 3$

$\therefore 3 = 1 - 2 + b$ $\qquad \therefore b = 4$

$\therefore ab = -2 \times 4 = -8$

---

4 (b)

Solution :

$\because y = \ln(\tan x)$ $\qquad \therefore y' = \frac{\sec^2 x}{\tan x}$

, $\because$ the x-coordinate of the tangency point $= \frac{\pi}{4}$

, $\because f\left(\frac{\pi}{4}\right) = \ln \tan\left(\frac{\pi}{4}\right) = 0$

$\therefore$ The tangency point $= \left(\frac{\pi}{4} , 0\right)$

$\therefore [y']_{x = \frac{\pi}{4}} = 2$

$\therefore$ The slope of normal $= \frac{-1}{2}$

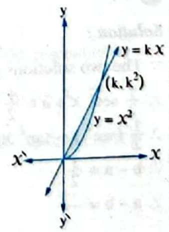
$\therefore$ The equation of normal $= \frac{y - 0}{x - \frac{\pi}{4}} = \frac{-1}{2}$

i.e. $8y + 4x = \pi$

---

5 (b)

Solution :



Putting $x^2 = kx$

$\therefore x^2 - kx = 0$

$\therefore x(x - k) = 0$

$\therefore x = 0$ or $x = k$

$\therefore$ The point of intersections are $(0 , 0)$, (k , k^2)

$\therefore \int_0^k (kx - x^2)\,dx = \frac{9}{2}$

$\therefore \left[\frac{1}{2}kx^2 - \frac{1}{3}x^3\right]_0^k = \frac{9}{2}$

$\therefore \frac{1}{2}k^3 - \frac{1}{3}k^3 = \frac{9}{2}$

$\therefore \frac{1}{6}k^3 = \frac{9}{2}$

$\therefore k^3 = 27$ $\qquad \therefore k = 3$

---

6 (b)

Solution :

$\because \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

$\therefore \frac{1}{2} = \frac{dy}{dt} \div (-3)$ $\qquad \therefore \frac{dy}{dt} = \frac{-3}{2}$

---

7 (a)

Solution :

$\because f'(x) = 0$ at $x = 2$ and its sign changes from positive to negative

$\therefore$ The function f has a local maximum value at $x = 2$

$\therefore$ (b) is true

, $\because$ the slope of the drawn curve is negative for each $x < 4$

$\therefore f''(x)$ is negative for each $x < 4$

, $\because$ the slope of the drawn curve is positive for each $x > 4$

$\therefore f''(x)$ is positive for each $x > 4$

$\therefore$ The curve of the function is convex upwards in $]-\infty , 4[$

, convex downward in $]4 , \infty[$

$\therefore$ (c) is true

, $\because f'(x)$ is positive for each $x < 2$

$\therefore$ The function f is increasing for each $x < 2$

, $\because -2 > -3$ $\qquad \therefore f(-2) > f(-3)$

$\therefore$ (d) is true

, $\because$ at $x = 4$ the function has no tangent

$\therefore$ There's no inflection point in spite of the changing of the convexity of the curve at $x = 4$

$\therefore$ (a) is wrong.

8 (b)

Solution :

$\because x = \tan(1+y)$ by differentiate with respect to x

$\therefore 1 = \sec^2(1+y)\,\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{1}{\sec^2(1+y)} = \frac{1}{1+\tan^2(1+y)} = \frac{1}{1+x^2}$

at $x = 2$

$\frac{dy}{dx} = 0.2$

9 (b)

10 (a)

Solution :

$\because y = e^{x^3-4}$ $\qquad \therefore \dot{y} = 3x^2 e^{x^3-4}$

$\therefore 2\dot{y} = 6x^2 y$ $\qquad \therefore m = 6$

11 (b)

Solution :

Let the coordinate of the point A is $\left(x , \frac{4}{x^2}\right)$

$\therefore$ The perimeter of the rectangle (P) $= 2\left(x + \frac{4}{x^2}\right)$

$\therefore \acute{P} = 2(1 - 8x^{-3})$

, put $\acute{P} = 0$ $\qquad \therefore 1 = 8x^{-3}$

$\therefore x = 2$

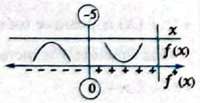
, $\ddot{P}(2) = 3$

$\therefore$ The perimeter (P) has minimum value at $x = 2$

$\therefore$ The coordinate of A is (2 , 1) in which the perimeter has minimum value.

12 (c)

Solution :



$\because f(x) = e^x(x+3)$

$\therefore \acute{f}(x) = e^x(x+3) + e^x$

$= e^x(x+4)$

$\therefore \ddot{f}(x) = e^x(x+4) + e^x = e^x(x+5)$

, put $\ddot{f}(x) = 0$

$\therefore e^x(x+5) = 0$ $\qquad \therefore x = -5$

$\therefore$ The curve of the function is convex downwards on $]-5 , \infty[$

13 (d)

Solution :

$\int \frac{4x^3 - 4x}{x^4 - 2x^2 + 3}\,dx = \ln|x^4 - 2x^2 + 3| + c$

$\therefore a = 4$

14 (b)

Solution :

$\because$ Slope of tangent $= \frac{dy}{dx} = e^x - e^{-x}$

$\therefore y = \int (e^x - e^{-x})\,dx = e^x + e^{-x} + c$

$\because$ (0 , 4) satisfies the equation of the curve

$\therefore 4 = 2 + c$ $\qquad \therefore c = 2$

equation of the curve :

$y = e^x + e^{-x} + 2$

15 (a)

Solution :

The length of the side of the smaller square after time "t" is $x = (1+t)$ cm.

The length of the side of the bigger square after time "t" is $y = \left(4 - \frac{1}{2}t\right)$ cm.

$\therefore$ The area (A) $= \frac{1}{2}\sqrt{2}\,x \times \frac{1}{2}\sqrt{2}\,y - \frac{1}{2}x^2$

$= \frac{1}{2}(xy - x^2)$

$\therefore A = \frac{1}{2}\left[(1+t)\left(4 - \frac{1}{2}t\right) - (1+t)^2\right]$

$= \frac{3}{4}(2 + t - t^2)$

$\therefore \frac{dA}{dt} = \frac{3}{4}(1 - 2t)$ $\qquad$ put $\frac{dA}{dt} = 0$ $\qquad t = \frac{1}{2}$

, $\because \frac{d^2y}{dt^2} = -\frac{3}{2}$ (negative)

$\therefore$ A stopped increasing instantaneously at $t = \frac{1}{2}$

16 (b)

Solution :

$\because$ The two solutions are correct

$\therefore \frac{1}{2}\sec^2 x + a = \frac{1}{2}\tan^2 x + b$

$\therefore \frac{1}{2}(\sec^2 x - \tan^2 x) = b - a$

$\therefore b - a = \frac{1}{2}$

$\therefore a - b = -\frac{1}{2}$

17 (a)

Solution :

$\int \frac{2}{y}\,dy = \int \frac{1}{x}\,dx$

$\therefore 2\ln|y| = \ln|x| + c$

$\therefore \ln y^2 = \ln|x| + c$

18 (b)

Solution :

$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{e^{\cot x}}{\sin^2 x}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^{\cot x} . \csc^2 x\,dx$

$= \left[-e^{\cot x}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -e^0 + e = e - 1$

19 (c)

Solution :

$\because y = a(1 - \cos\theta)$ $\qquad \therefore \frac{dy}{d\theta} = a\sin\theta$

, $\because x = a(\theta + \sin\theta)$ $\qquad \therefore \frac{dx}{d\theta} = a(1 + \cos\theta)$

$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)}$

$= \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1 + 2\cos^2\frac{\theta}{2} - 1} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$

20 (c)

Solution :

$\int \frac{e^{2x}}{e^{-x}}\,dx = \int e^{3x}\,dx = \frac{1}{3}e^{3x} + c = \frac{1}{3}\frac{e^{2x}}{e^{-x}} + c$

21 (d)

Solution :

$y = (\cos x)^{\cos x}$

(by taking logarithm of both sides to the base e)

$\therefore \ln y = \cos x \ln\cos x$

(by differentiating both sides with respect to x)

$\therefore \frac{\dot{y}}{y} = \cancel{\cos} x\left(\frac{-\sin x}{\cancel{\cos} x}\right) - \sin x \ln\cos x$

$\therefore \acute{f}(x) = (\cos x)^{\cos x}\left[-\sin x - \sin x \ln\cos x\right]$

$\therefore \acute{f}(0) = 0$

22 (d)

Solution :

$2x+3$	$\ln x$
x^2+3x	$\frac{1}{x}$

$\int (2x+3)\ln x\,dx$

$= (x^2 + 3x)\ln x - \int (x+3)\,dx$

$\therefore yz = x(x+3)\ln x$

23 (d)

24 (d)

Solution :

$y = (f \circ g)(x) = f(g(x)) = \frac{2}{3x+1}$

$\therefore \frac{dy}{dx} = \frac{-2 \times 3}{(3x+1)^2}$ $\qquad \therefore \left(\frac{dy}{dx}\right)_{x=-2} = \frac{-6}{25}$

25 (c)

Solution :

Volume $= \pi\int_k^{2k} y^2\,dx$

$= \pi\int_k^{2k}\left(\frac{1}{x}\right)^2 dx = \pi\left[-\frac{1}{x}\right]_k^{2k}$

$= \pi\left[-\frac{1}{2k} + \frac{1}{k}\right] = \frac{\pi}{2k}$

$\therefore \frac{\pi}{2k} = \frac{\pi}{3}$ $\qquad \therefore 2k = 3$ $\qquad k = \frac{3}{2}$

Exam 4

1 (d)

Solution :

$\lim_{x \to 0} \frac{e^{x+a} - e^a}{x} = \lim_{x \to 0} e^a\left(\frac{e^x - 1}{x}\right)$

$= e^a \times \lim_{x \to 0}\frac{e^x - 1}{x} = e^a$

$\therefore e^a = \frac{1}{e} = e^{-1}$ $\qquad \therefore a = -1$

2 (c)

3 (a)

4 (b)

Solution :

$\int_0^1 \left[a\sqrt{x} - (ax - 2)\right]dx = \frac{13}{6}$

$\therefore \left[\frac{2}{3}a x^{\frac{3}{2}} - \frac{1}{2}a x^2 + 2x\right]_0^1 = \frac{13}{6}$

$\therefore \frac{2}{3}a - \frac{1}{2}a + 2 = \frac{13}{6}$

$\therefore \frac{1}{6}a = \frac{1}{6}$ $\qquad \therefore a = 1$

5 (c)

Solution :

Let $z=\frac{1}{y}=(x+1)^2$, $k=x^2$

$\therefore \frac{dz}{dk}=\frac{dz}{dx}\div\frac{dk}{dx}=\frac{2(x+1)}{2x}=1+\frac{1}{x}$

at $x=1$ $\qquad \therefore \frac{dz}{dk}=2$

---

6 (a)

---

7 (b)

Solution :

$f(x)=x\ln x$

$\therefore f'(x)=x\times\frac{1}{x}+\ln x=1+\ln x$

, put $f'(x)=0$ $\qquad \therefore \ln x=-1$

$\therefore x=e^{-1}$

$\therefore f(e)=e$, $f(e^{-1})=-e^{-1}$, $f(e^{-2})=-2e^{-2}$

$\therefore$ The absolute maximum value $=e$

---

8 (d)

Solution :

$\int\sqrt{x}\left(1+\sqrt{x}\right)dx=\int\left(x^{\frac{1}{2}}+x\right)dx$

$=\frac{2}{3}x^{\frac{3}{2}}+\frac{1}{2}x^2+c$

---

9 (c)

Solution :

$\int\frac{2}{x\ln x^2}dx=\int\frac{2}{2x\ln|x|}dx=\int\frac{\frac{1}{x}}{\ln|x|}dx$

$=\ln\left|\ln|x|\right|+c$

---

10 (d)

Solution :

$e^{xy}-x^2+y^3=0 \qquad (1)$

at $x=0$

$\therefore 1-0+y^3=0$ $\qquad \therefore y^3=-1$ $\qquad \therefore y=-1$

By differentiating equation (1) with respect to x

$\therefore e^{xy}(y+xy')-2x+3y^2y'=0$

, at $(0,-1)$

$\therefore 1(-1+0)-0+3(-1)^2y'=0$

$\therefore -1+3y'=0$ $\qquad \therefore y'=\frac{1}{3}$

$\therefore \left(\frac{dy}{dx}\right)_{x=0}=\frac{1}{3}$

11 (d)

Solution :

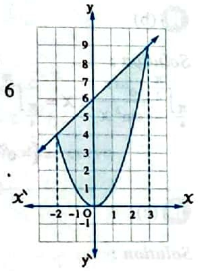
Equation of the straight line which passes through the two points $(0,6)$, $(1,7)$:

$\frac{y-6}{x-0}=\frac{7-6}{1-0}$

$\therefore$ Equation of the straight line is

$y=x+6$

to find the intersection points with the curve put : $x^2=x+6$



$\therefore x^2-x-6=0$

$\therefore (x-3)(x+2)=0$

$\therefore x=3$ or $x=-2$

$\therefore$ The volume

$=\pi\int_{-2}^{3}\left((x+6)^2-x^4\right)dx$

$=\pi\int_{-2}^{3}(x^2+36+12x-x^4)dx$

$=\pi\left[\frac{1}{3}x^3+36x+6x^2-\frac{1}{5}x^5\right]_{-2}^{3}$

$=\frac{500}{3}\pi$ cubic unit

---

12 (a)

Solution :

$\int\tan^2 x\,dx=\int(\sec^2 x-1)\,dx=\tan x-x+c$

---

13 (b)

$\int_{-3}^{3}(f'(x)-f(x))\,dx$

$=\int_{-3}^{3}f'(x)\,dx-\int_{-3}^{3}f(x)\,dx$

$=\left[f(x)\right]_{-3}^{3}-2\int_{0}^{3}f(x)\,dx$

$=f(3)-f(-3)-2\times5=f(3)-f(3)-10$

$=-10$ (where f is even)

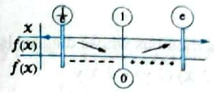
---

14 (c)

Solution :

$f(x)=1-\frac{1}{x}=\frac{x-1}{x}$

, let $f'(x)=0$ $\qquad \therefore x=1$



$\therefore$ The function is decreasing on $\left]\frac{1}{e},1\right[$

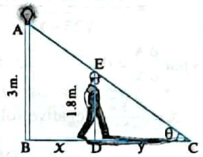
and is increasing on $]1,e[$

---

15 (d)

Solution :

From the similarity of two triangles ABC , EDC



$\therefore \frac{y}{y+x}=\frac{1.8}{3}=\frac{3}{5}$

$\therefore 3x=2y$

(by differentiation with respect to t)

$\therefore 3\frac{dx}{dt}=2\frac{dy}{dt}$

$\therefore \frac{dy}{dt}=\frac{3}{2}\times1.2=1.8$ m./sec.

(rate of change of the length of man's shadow)

---

16 (c)

Solution :

$f(x)=\log_x e=\frac{1}{\ln x}=(\ln x)^{-1}$

$\therefore f'(x)=-\frac{(\ln x)^{-2}}{x}=-\frac{(\log_x e)^2}{x}$

---

17 (a)

Solution :

$\because 3y^2\frac{dy}{dx}=2x+1$

$\therefore \int 3y^2\,dy=\int(2x+1)\,dx$

$\therefore y^3=x^2+x+c$

$\because x=1$ at $y=-1$

$\therefore (-1)^3=1+1+c$

$\therefore c=-3$

$\therefore$ The relation is : $y^3=x^2+x-3$

18 (b)

Solution :

$\because x=e^t$ $\qquad \therefore \frac{dx}{dt}=e^t$

$\because y=e^{2t+2}$ $\qquad \therefore \frac{dy}{dx}=2e^{2t+2}$

$\therefore \frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=\frac{2e^{2t+2}}{e^t}=2e^{t+2}$

at $x=1$ then $t=0$

Slope of tangent at $(x=1)$

is $\left(\frac{dy}{dx}\right)_{x=1}=2e^2$

$\therefore$ Slope of normal $=\frac{-1}{2e^2}$

---

19 (b)

Solution :

$\frac{dy}{dx}=-\sin x$, $\frac{d^2y}{dx^2}=-\cos x$

$\therefore \left(\frac{dy}{dx}\right)^2-y\frac{d^2y}{dx^2}$

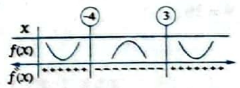
$=(-\sin x)^2-\cos x(-\cos x)$

$=\sin^2 x+\cos^2 x=1$

---

20 (b)

Solution :



The curve is convex upward in $]-4,3[$

---

21 (d)

Solution :

$\because f(x)=x^{\sin x}$

(by taking logarithm of both sides to the base e)

$\therefore \ln\left(f(x)\right)=\sin x\ln x$

(by differentiating both sides with respect to x)

$\therefore \frac{f'(x)}{f(x)}=\sin x\times\frac{1}{x}+\ln x\times\cos x$

$\therefore f'(x)=x^{\sin x}\left(\frac{\sin x}{x}+\cos x\ln x\right)$

$\therefore f'\left(\frac{\pi}{2}\right)=\frac{\pi}{2}\left(\frac{2}{\pi}+\text{zero}\right)=1$

22 (b)

Solution :

$\because {}_a\int^b \frac{d}{dx}(3x^2-2x)\,dx=0$

$\therefore [3x^2-2x]_a^b=0$

$\therefore (3b^2-2b)-(3a^2-2a)=0$

$\therefore 3(b^2-a^2)-2(b-a)=0$

$\therefore 3(b-a)(b+a)-2(b-a)=0$

$\therefore (b-a)(3b+3a-2)=0$

$\therefore b=a$ (refused) or $3b+3a=2$

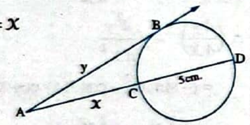
$\therefore a+b=\frac{2}{3}$

---

23 (c)

Solution :

Let AB = y , AC = x



$\therefore y^2=x(x+5)$

$\therefore y^2=x^2+5x$

$\therefore 2y\frac{dy}{dt}=(2x+5)\frac{dx}{dt}$ (1)

, at $x=4$

$\therefore y^2=4\times 9=36$

$\therefore y=6$

From (1) :

$2\times 6\times 1=(2\times 4+5)\times\frac{dx}{dt}$

$\therefore \frac{dx}{dt}=\frac{12}{13}$ cm./min.

---

24 (b)

Solution :

$\dot{f}(x)$ is increasing in $]-1,0[$

$\therefore \ddot{f}(x)$ is positive in $]-1,0[$

, $\because \dot{f}(x)$ is decreasing in $]0,1[$

$\therefore \ddot{f}(x)$ is negative in $]-1,0[$

$\therefore$ The point $(0, f(0))$ is inflection point.

25 (a)

Solution :

The figure is a rhombus whose diagonals $2x$, $2y$

$\therefore$ Area $=\frac{1}{2}(2x)(2y)=2xy$

$\because$ AB = 5 cm. $\quad\therefore y=\sqrt{25-x^2}$

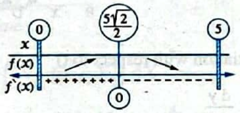
$\therefore A=2x\sqrt{25-x^2}$

$\therefore \frac{dA}{dx}=2\sqrt{25-x^2}+2x\times\frac{-\cancel{2}x}{\cancel{2}\sqrt{25-x^2}}$

$=\frac{2(25-x^2)-2x^2}{\sqrt{25-x^2}}=\frac{50-4x^2}{\sqrt{25-x^2}}$

, put $\frac{dA}{dx}=0$ $\quad\therefore 50-4x^2=0$

$\therefore x=\frac{5\sqrt{2}}{2}$ (negative solution is refused)



$\therefore$ A is maximum at $x=\frac{5\sqrt{2}}{2}$, $y=\frac{5\sqrt{2}}{2}$

i.e. $x=y$

Exam 5

1 (b)

Solution :

$\lim\limits_{x\to\infty}\left(\frac{x+a}{x+b}\right)^x=\lim\limits_{x\to\infty}\left(\frac{1+\frac{a}{x}}{1+\frac{b}{x}}\right)^x$

$=\lim\limits_{x\to\infty}\frac{\left(1+\frac{a}{x}\right)^x}{\left(1+\frac{b}{x}\right)^x}=\frac{e^a}{e^b}=e^{a-b}$

$\therefore c=a-b$

---

2 (a)

---

3 (d)

Solution :

$\because \ddot{f}(x)>0$ for each $x\neq 3$

$\therefore$ The curve is convex downwards for each $x\neq 3$

, $\because \ddot{f}(3)$ undefined

$\therefore$ The figure (d) can represent the curve of the function f

4 (a)

Solution :

Area $={}_{-1}\int^2 y\,dx={}_{-1}\int^2(3x^2+4)\,dx$

$=[x^3+4x]_{-1}^2$

$=[(2)^3+4(2)]-[(-1)^3+4(-1)]$

$=21$ square unit.

---

5 (c)

Solution :

$y=\sqrt{2(\sec x+\tan x)}$

$\frac{dy}{dx}=\frac{2(\sec x\tan x+\sec^2 x)}{2\sqrt{2(\sec x+\tan x)}}=\frac{\sec x(\tan x+\sec x)}{\sqrt{2(\sec x+\tan x)}}$

$\therefore \frac{1}{y}\left(\frac{dy}{dx}\right)=\frac{\sec x(\tan x+\sec x)}{2(\sec x+\tan x)}=\frac{1}{2}\sec x$

---

6 (c)

Solution :

Let (x , y) is a point on the curve

$\therefore y^2=4x$ $\quad\therefore x=\frac{1}{4}y^2$

$\therefore$ The point $\left(\frac{1}{4}y^2 , y\right)$ lies on the curve

$\therefore$ The distance between the point and the straight line.

$s=\frac{|x_1-2y_1+10|}{\sqrt{1^2+(-2)^2}}=\frac{|\frac{1}{4}y^2-2y+10|}{\sqrt{5}}$

$\therefore s=\frac{1}{\sqrt{5}}|\frac{1}{4}y^2-2y+10|$

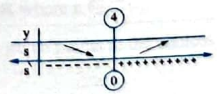
$\because \frac{1}{4}y^2-2y+10$ is positive for all values of y

(for the discriminant < 0)

$\therefore s=\frac{1}{\sqrt{5}}\left(\frac{1}{4}y^2-2y+10\right)$

$\therefore \dot{s}=\frac{1}{\sqrt{5}}\left(\frac{1}{2}y-2\right)$

Put $\dot{s}=0$ $\quad\therefore y=4$



The distance is smallest at y = 4

$\therefore$ Smallest distance is $s=\frac{6\sqrt{5}}{5}$ length unit.

7 (b)

Solution :

$\because x\frac{dy}{dx}-y=3$

$\therefore x\frac{dy}{dx}=y+3$

$\therefore \frac{dy}{dx}=\frac{y+3}{x}$

$\therefore \int\frac{dy}{y+3}=\int\frac{dx}{x}$

$\therefore \ln|y+3|=\ln|x|+c$

at $x=1$, $y=-2$ $\quad\therefore c=0$

$\therefore \ln|y+3|=\ln|x|$

i.e. $|y+3|=|x|$

---

8 (b)

Solution :

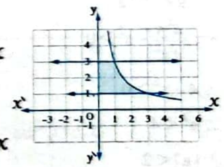
$\because \int\frac{2e^x+1}{e^x}dx=\int(2+e^{-x})\,dx=2x-e^{-x}+c$

---

9 (a)

Solution :

$y=\frac{3}{x}=3x^{-1}$



$\therefore v=\pi\,{}_0\int^1[3^2-1^2]\,dx$

$+\pi\,{}_1\int^3(3x^{-1})^2$

$-(1)^2\,dx=\pi\,{}_0\int^1 8\,dx$

$+\pi\,{}_1\int^3(9x^{-2}-1)\,dx$

$=\pi[8x]_0^1+\pi[-9x^{-1}-x]_1^3$

$=8\pi+4\pi=12\pi$ cubic unit.

---

10 (c)

Solution :

$\because f(x)=cx^2+g(x)$

$\therefore \dot{f}(x)=2cx+\dot{g}(x)$

$,\ddot{f}(x)=2c+\ddot{g}(x)$

, $\because$ (1 , 5) is an inflection point to the curve of the function f

$\therefore \grave{f}(1)=0$ $\therefore 2c+\grave{g}(1)=0$

$\therefore 2c=-6$ $\therefore c=-3$

$, f(1)=5$ $\therefore 5=c+g(1)$

$\therefore c+k=5$ $\therefore -3+k=5$

$\therefore k=8$ $\therefore k-c=8-(-3)=11$

11 (a)

Solution :

$$f(x)=-|x|+1=\begin{cases}-x+1 & , x\ge 0\\ x+1 & , x<0\end{cases}$$

$$\therefore \grave{f}(x)=\begin{cases}-1 & , x>0\\ 1 & , x<0\end{cases}$$

$\therefore$ The function is decreasing in $]0, \infty[$

12 (a)

Solution :

$$\frac{dy}{dz}=\frac{dy}{dx}\div\frac{dz}{dx}=\frac{\tan x}{\cot x}=\tan^2 x$$

$$\therefore \frac{d^2y}{dz^2}=2\tan x\sec^2 x\frac{dx}{dz}=2\tan^2 x\sec^2 x$$

$$\therefore \left(\frac{d^2y}{dz^2}\right)_{x=\frac{\pi}{4}}=2\times 1\times 2=4$$

$$, \because \frac{d^2y}{dz^2}-a^2>0$$

$\therefore 4-a^2>0$

$\therefore a^2<4$

$\therefore |a|<2$

$\therefore -2<a<2$

$\therefore a\in]-2, 2[$

13 (c)

Solution :

$\Delta ADE \sim \Delta ACB$

$$\therefore \frac{AD}{AC}=\frac{AE}{AB}$$

$$\therefore \frac{1+\tan^2 x}{y}=\frac{3}{3+\tan^2 x}$$

$$\therefore \frac{\sec^2 x}{y}=\frac{3}{2+1+\tan^2 x}$$

$$\therefore y=\frac{1}{3}\sec^2 x(2+1+\tan^2 x)$$

$$=\frac{2}{3}\sec^2 x+\frac{1}{3}\sec^2 x(1+\tan^2 x)$$

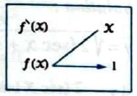
$$\therefore y=\sec^2 x+\frac{1}{3}\sec^2 x\tan^2 x$$

$$\therefore \int y\,dx=\int\left(\sec^2 x+\frac{1}{3}\sec^2 x\tan^2 x\right)dx$$

$$=\tan x+\frac{1}{9}\tan^3 x+c$$

14 (a)

Solution :



$$\int_0^4 x\grave{f}(x)\,dx$$

$$=[x f(x)]_0^4-\int_0^4 f(x)\,dx$$

$$=4f(4)-8$$

$$=4\times-3-8=-20$$

15 (a)

Solution :

$$\frac{dy}{dx}=\frac{\cos x}{\sin x}=\cot x$$

$$\therefore \frac{d^2y}{dx^2}=-\csc^2 x$$

16 (a)

Solution :

Let the side length of the hexagon $= x$

$\therefore$ The shaded area (A) $=3\times\frac{1}{2}x^2\sin 120°$

$$\therefore A=\frac{3\sqrt{3}}{4}x^2$$

$$\therefore \frac{dA}{dt}=\frac{3\sqrt{3}}{2}x\frac{dx}{dt}$$

at $x=6$

$$\frac{dA}{dt}=\frac{3\sqrt{3}}{2}\times 6\times\sqrt{3}=27\text{ cm}^2\text{/sec.}$$

17 (c)

Solution :

$$f(x)=x^3+3x^2-9x$$

$$\therefore \grave{f}(x)=3x^2+6x-9$$

Put $\grave{f}(x)=0$

$\therefore x=-3\notin[-2, 2]$ or $x=1$

$, f(-2)=22 \quad , f(2)=2 \quad , f(1)=-5$

$\therefore$ The absolute minimum value $=a=-5$

The absolute maximum value $=b=22$

$\therefore b-a=22+5=27$

18 (b)

Solution :

$\because x^2=e^{2y}$ (differentiate with respect to x)

$$\therefore 2x=2e^{2y}\frac{dy}{dx} \qquad \therefore \frac{dy}{dx}=\frac{x}{e^{2y}}$$

$$\therefore x\frac{dy}{dx}=\frac{x^2}{e^{2y}}=1$$

19 (d)

Solution :

$y^2=8x$

$$\therefore 2y\frac{dy}{dx}=8 \qquad \therefore \frac{dy}{dx}=\frac{4}{y}$$

$$, \frac{dx}{dy}=\frac{y}{4}$$

$$\therefore \frac{4}{y}=\frac{y}{4} \qquad \therefore y^2=16$$

$\therefore y=4$ or $y=-4$ (refused)

, at $y=4$, then $x=2$

$\therefore$ The point is $(2, 4)$

20 (c)

Solution :

$\because 2x+1=\sin y$ (differentiate with respect to x)

$$\therefore 2=\cos y\frac{dy}{dx} \qquad \therefore \frac{dy}{dx}=\frac{2}{\cos y}$$

$, \because$ the tangent // y-axis

$\therefore$ Denominator $=0$ $\therefore \cos y=0$

$\therefore y=\frac{\pi}{2}+n\pi$ where $n\in\mathbb{Z}$

$, \because$ in all the given points in the choices $y\in[0, \pi]$

$$\therefore y=\frac{\pi}{2}$$

$\therefore 2x+1=\sin\frac{\pi}{2}$, then $x=0$

$\therefore$ The point $\left(0, \frac{\pi}{2}\right)$

21 (d)

Solution :

$g(x)=f(3-x)$

$\therefore \grave{g}(x)=-\grave{f}(3-x)$

$\therefore \grave{g}(1)=-\grave{f}(2)=-5$

22 (d)

23 (d)

Solution :

$\grave{f}(x)=3ax^2$

$\grave{f}(1)=3a$

$, \grave{g}(x)=2b\sin x\cos x=b\sin 2x$

$$\therefore \grave{g}\left(\frac{\pi}{4}\right)=b$$

$$\therefore 3a=\frac{-1}{b}$$

$\therefore 3ab=-1$

$$\therefore ab=\frac{-1}{3}$$

24 (b)

Solution :

$$\int(\sin x+\cot x)^9(\cos x-\csc^2 x)\,dx$$

$$=\frac{1}{10}(\sin x+\cot x)^{10}+c$$

25 (c)

Solution :

$\because$ The function is even , $\int_{-5}^{5} f(x)\,dx=2a$

$$\therefore \int_0^5 f(x)\,dx=\int_{-5}^0 f(x)\,dx=a$$

$$, \int_0^3 f(x)\,dx=\int_{-3}^0 f(x)\,dx=b$$

$$\therefore \int_{-5}^{-3} f(x)\,dx=\int_{-5}^0 f(x)\,dx-\int_{-3}^0 f(x)\,dx$$

$$=a-b$$

Exam 6

1 (d)

2 (a)

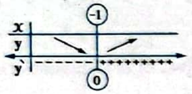
Solution :

$\frac{dy}{dx} = e^x + x e^x = e^x (1 + x)$

Put $\frac{dy}{dx} = 0$

$\therefore e^x = 0$ (refused)

or $1 + x = 0$ $\quad\therefore x = -1$



$\therefore$ The function has local minimum value at $x = -1$

3 (d)

Solution :

$\frac{dy}{dx} = 6x$

$\therefore \left(\frac{dy}{dx}\right)_{x=1} = 6$

$\therefore$ Slope of the tangent $= 6$

$\therefore$ The equation of the tangent : $\frac{y+2}{x-1} = 6$

i.e. $6x - y - 8 = 0$

by checking the four choices we get that the point $(0, -8)$ which is satisfying the equation.

4 (d)

Solution :

The perimeter of the circular sector $p = 2r + \ell$

$\therefore \ell = p - 2r$

$\therefore$ its area $(A) = \frac{1}{2} \ell r = \frac{1}{2}(p - 2r) r = \frac{1}{2} pr - r^2$

$\therefore \frac{dA}{dr} = \frac{1}{2} p - 2r$ $\quad\therefore \frac{d^2A}{dr^2} = -2 < 0$

means it has local maximum

Put $\frac{dA}{dr} =$ zero $\quad\therefore \frac{1}{2} p = 2r$

$\therefore r = \frac{1}{4} p$

5 (c)

Solution :

$x = e^{2t}$ $\quad\therefore \frac{dx}{dt} = 2e^{2t}$

, $y = t^3$ $\quad\therefore \frac{dy}{dt} = 3t^2$

$\therefore \frac{dy}{dx} = \frac{3t^2}{2e^{2t}}$

$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{3t^2}{2e^{2t}}\right) = \left(\frac{2e^{2t}(6t) - 3t^2 \times 4e^{2t}}{4e^{4t}}\right) \times \frac{dt}{dx}$

$= \frac{12te^{2t}(1-t)}{4e^{4t}} \times \frac{1}{2e^{2t}}$

$= \frac{3}{2} t e^{-4t}(1-t)$, at $t = 1$

$\therefore \frac{d^2y}{dx^2} =$ zero

6 (d)

Solution :

$\because \int_{-4}^{4} f(x)\, dx = 20$

, f is even

$\therefore \int_{-4}^{0} f(x)\, dx = 10$

$\therefore \int_{-4}^{2} f(x)\, dx = \int_{-4}^{0} f(x)\, dx + \int_{0}^{2} f(x)\, dx$

$= 10 + 6 = 16$

7 (c)

Solution :

$f(x) = \cot x$

$\therefore f'(x) = -\csc^2 x$

$\therefore f''(x) = -2\csc x\,[-\csc x \times \cot x]$

$= 2(\csc x)^2 \cot x$

$\therefore f''\left(\frac{\pi}{4}\right) = 2 \times 2 \times 1 = 4$

8 (d)

Solution :

$\because \int_0^5 f(x)\, dx = \int_0^3 3x^2\, dx + \int_3^5 (2x + 1)\, dx$

$= [x^3]_0^3 + [x^2 + x]_3^5$

$= 27 + 30 - 12 = 45$

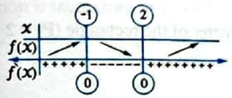
9 (d)

Solution :

$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$

$= 6(x + 1)(x - 2)$

$\therefore f'(x) = 0$ $\quad$ when $x = -1$ or $x = 2$



there exist a local maximum value $= f(-1) = 19$

10 (c)

Solution :

Area of the triangle $(A) = \frac{1}{2} \times a \times a \times \sin 60° = \frac{\sqrt{3}}{4} a^2$

$\therefore \frac{dA}{dt} = \frac{\sqrt{3}}{2} a \frac{da}{dt} = \frac{\sqrt{3}}{2} a k$

11 (d)

Solution :

by checking the choices

(1) $f(x) = 3 - x^2$ $\quad\therefore f'(x) = -2x$

$\therefore f''(x) = -2$

i.e. Convex upwards

(2) $f(x) = 3 - x^3$

$\therefore f'(x) = -3x^2$ $\quad\therefore f''(x) = -6x$

$\therefore$ it is convex downwards at $x \in]-\infty, 0[$

and convex upwards at $x \in]0, \infty[$

(3) $f(x) = 3 - x^4$, $f'(x) = -4x^3$

$\therefore f''(x) = -12$ means it is convex upwards

(4) $f(x) = 3 + x^4$

$\therefore f'(x) = 4x^3$ $\quad\therefore f''(x) = 12x^2$

means it is convex downwards in $\mathbb{R}$

i.e. (d) is the correct choice.

12 (a)

Solution :

$\int \tan\theta\, d\theta = -\int \frac{-\sin\theta}{\cos\theta}\, d\theta = -\ln|\cos\theta| + c$

13 (b)

Solution :

Volume (v) $= \pi \int_1^4 (\sqrt{x})^2\, dx = \pi\left[\frac{1}{2} x^2\right]_1^4$

$= \pi\left[8 - \frac{1}{2}\right] = \frac{15}{2}\pi$ cubic unit.

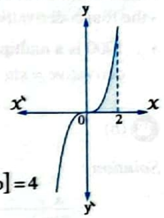
14 (a)

Solution :

The area $= \int_0^2 y\, dx$

$= \int_0^2 x^3\, dx$

$= \left[\frac{x^4}{4}\right]_0^2 = \left[\frac{16}{4}\right] - [\text{zero}] = 4$



15 (c)

Solution :

$\cos^4 x + \sin^4 x = (\cos^2 x + \sin^2 x)^2 - 2\cos^2 x \sin^2 x$

$= 1 - \frac{1}{2} \times 4\cos^2 x \sin^2 x$

$= 1 - \frac{1}{2}(\sin 2x)^2$

$\therefore \frac{d}{dx}(\cos^4 x + \sin^4 x)$

$= -\frac{1}{2} \times 2\sin 2x \times \cos 2x \times 2$

$= -2\sin 2x \cos 2x = -\sin 4x$

$\therefore \frac{d^2}{dx^2}(\cos^4 x + \sin^4 x) = -4\cos 4x$

16 (c)

Solution :

$\because$ The area of the sphere $A = 4\pi r^2$

$\therefore \frac{dA}{dt} = 8\pi r \frac{dr}{dt}$

when $r = 30$ cm. : $\therefore 6 = 8\pi \times 30 \times \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{1}{40\pi}$ cm./sec.

$\therefore$ The volume of the sphere $v = \frac{4}{3}\pi r^3$

$\therefore \frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\therefore \frac{dv}{dt} = 4\pi \times (30)^2 \times \frac{1}{40\pi} = 90$ cm³/sec.

17 (a)

Solution :

$f(x) = \sin x$

The first derivative $= \cos x$

, the second derivative $= -\sin x$

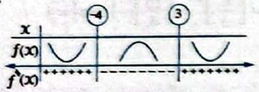
, the third derivative $= -\cos x$

, the fourth derivative $= \sin x$

, $\because$ 1000 is a multiple of 4 , then the 1000th derivative $= \sin x$

18 (b)

Solution :



The curve is convex upwards in $]-4, 3[$

19 (a)

Solution :

$\because$ The slope of the perpendicular $= \frac{-2y}{x}$

$\therefore$ The slope of the tangent $= \frac{x}{2y}$

$\therefore \frac{dy}{dx} = \frac{x}{2y} \qquad \therefore 2y\,dy = x\,dx$

$\therefore \int 2y\,dy = \int x\,dx \quad \therefore y^2 = \frac{1}{2}x^2 + c$

, $(0, 3)$ lies on the curve :

$\therefore 9 = \frac{1}{2}(0)^2 + c \qquad \therefore c = 9$

$\therefore$ The equation of the curve is : $y^2 = \frac{1}{2}x^2 + 9$

$\therefore 2y^2 = x^2 + 18$

20 (b)

Solution :

$$f(x) = \begin{cases} x & , \ x \geq 0 \\ -x & , \ x < 0 \end{cases}$$

$$\therefore f'(x) = \begin{cases} 1 & , \ x > 0 \\ -1 & , \ x < 0 \end{cases}$$

$\therefore f'(-6) = -1$

21 (c)

22 (a)

23 (c)

Solution :

Let the perimeter of the rectangle (P) $= 2(x + y) = 14$

$\therefore x + y = 7$

, Area of the rectangle (A) $= x\,y$

$\therefore A = x(7 - x) = 7x - x^2$

$\therefore \frac{dA}{dx} = 7 - 2x \ , \ \frac{d^2A}{dx^2} = -2 < 0$ has local maximum

Put $\frac{dA}{dt} = 0 \ , \ x = \frac{7}{2}$

$\therefore$ The greatest area (A) $= 7 \times 3\frac{1}{2} - (3\frac{1}{2})^2 = 12.25 \text{ cm}^2$

24 (d)

Solution :

$$\int \frac{1}{1 - \cos^2 x}\,dx = \int \frac{1}{\sin^2 x}\,dx$$

$$= \int \csc^2 x\,dx = -\cot x + c$$

25 (a)

Solution :

$$\int_{-2}^{5} [3f(x) - 6x]\,dx = 3\int_{-2}^{5} f(x)\,dx - [3x^2]_{-2}^{5}$$

$$= 3\left[\int_{-2}^{3} f(x)\,dx + \int_{3}^{5} f(x)\,dx\right] - [3 \times 25 - 3 \times 4]$$

$$= 3[9 + (-4)] - 63 = -48$$

Exam 7

1 (c)

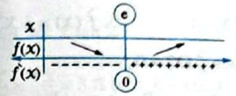
Solution :

$f(x) = \frac{x}{\ln x}$

$\therefore f'(x) = \frac{\ln x - x \times \frac{1}{x}}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$

when $f'(x) = 0 \qquad \therefore \ln x = 1 \quad \therefore x = e$

, $f'(x)$ is undefined $\qquad \therefore \ln x = 0$ (refused)



$\therefore$ The function is increasing in $]e, \infty[$

2 (b)

3 (a)

Solution :

$$\lim_{x \to \infty} \left(\frac{x + 6}{x + 2}\right)^{x+5} = \lim_{x \to \infty} \left(1 + \frac{4}{x + 2}\right)^{x+5}$$

Let $y = \frac{4}{x + 2}$ *i.e.* $x = \frac{4}{y} - 2$

in case $x \longrightarrow \infty \qquad \therefore y \longrightarrow 0$

$$\therefore \lim_{y \to 0} (1 + y)^{\frac{4}{y} - 2 + 5} = \lim_{y \to 0} (1 + y)^{\frac{4}{y} + 3}$$

$$= \lim_{y \to 0} \left[\left((1 + y)^{\frac{1}{y}}\right)^4 \times (1 + y)^3\right]$$

$$= e^4 \times 1 = e^4$$

4 (c)

Solution :

Let the dimensions of the rectangle are x and y and the area = A

$\therefore A = x\,y$

$\because x^2 + y^2 = 64 \qquad \therefore y^2 = 64 - x^2$

$\therefore y = \sqrt{64 - x^2} \qquad \therefore A = x\sqrt{64 - x^2}$

$$\therefore \frac{dA}{dx} = x \times \frac{-2x}{2\sqrt{64 - x^2}} + \sqrt{64 - x^2}$$

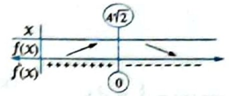
$$= \frac{-x^2}{\sqrt{64 - x^2}} + \sqrt{64 - x^2}$$

$$= \frac{-x^2 + 64 - x^2}{\sqrt{64 - x^2}} = \frac{64 - 2x^2}{\sqrt{64 - x^2}}$$

At the maximum or minimum values

$A' =$ zero

$\therefore 64 - 2x^2 = 0 \qquad \therefore x = 4\sqrt{2}$



$\because$ Sign of A' changes from positive to negative around $x = 4\sqrt{2}$

$\therefore$ At $x = 4\sqrt{2}$ the area A has maximum value

$\therefore$ The dimensions of the rectangle are $4\sqrt{2}, 4\sqrt{2}$

$\therefore$ The area $= 4\sqrt{2} \times 4\sqrt{2} = 32 \text{ cm}^2$

5 (d)

Solution :

$f(x) = 2x^3 + 3x^2 - 12x + 5$

$\therefore f'(x) = 6x^2 + 6x - 12$

$\therefore f''(x) = 12x + 6$

When $f''(x) = 0 \qquad \therefore x = \frac{-1}{2}$

$\because f''(x)$ changes its sign around of $x = \frac{-1}{2}$

$\therefore$ Their exists an inflection point at $x = \frac{-1}{2}$

6 (b)

Solution :

$f(x) = x$ is only odd function in the choices

$\therefore \int_{-2}^{2} x\,dx =$ zero

7 (c)

Solution :

$f(x) = \sin 2x \cos 2x = \frac{1}{2}\sin 4x$

$\therefore f'(x) = 2\cos 4x \quad \therefore f''(x) = -8\sin 4x$

$\therefore f''\left(\frac{\pi}{3}\right) = -8\sin\left(\frac{4\pi}{3}\right) = 4\sqrt{3}$

8 (a)

Solution :

$\because y = (\sin x)^{\tan x}$ (take ln of each side)

$\therefore \ln y = \tan x \ln(\sin x)$

(by differentiaion with respect to x)

$\therefore \frac{y'}{y} = \sec^2 x \ln(\sin x) + \tan x \times \frac{\cos x}{\sin x}$

$= \sec^2 x \ln(\sin x) + 1$

$\therefore \frac{dy}{dx} = (\sin(x))^{\tan x}(\sec^2 x \ln(\sin x) + 1)$

9 (d)

Solution :

volume of cone (v) $= \frac{1}{3}\pi r^2 h$

$, \because h = 2r$

$\therefore v = \frac{1}{3}\pi r^2 \times 2r = \frac{2\pi}{3} r^3$

$\therefore \frac{dv}{dt} = 2\pi r^2 \times \frac{dr}{dt} = 2\pi \times (5)^2 \times \frac{1}{\pi} = 50$ cm³/sec.

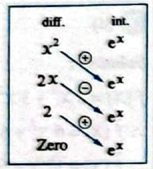
---

10 (b)

Solution :

$\therefore \int x^2 e^x \, dx$

$= x^2 e^x - 2x e^x + 2e^x + c$



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11 (c)

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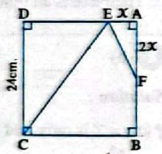
12 (a)

Solution :



Let AF = 2 AE = $2x$

$\therefore$ ED $= 24 - x$

Area of FBCE (A)

$= 24 \times 24 - \left[\left(\frac{1}{2}x \cdot 2x\right) + \left(\frac{1}{2} \times 24 \times (24 - x)\right)\right]$

$= 576 - (x^2 + 288 - 12x)$

$\therefore \frac{dA}{dx} = -2x + 12$, put $\frac{dA}{dx} = 0$

$\therefore x = 6$, $\frac{d^2A}{dx^2} < 0$

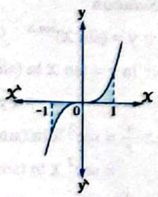
$\therefore$ The area is as great as possible at $x = 6$

$\therefore$ The greatest area of the shape = 324 cm².

---

13 (b)

Solution :



The required area

$= \left| \int_{-1}^{0} x^3 \, dx \right| + \int_{0}^{1} x^3 \, dx$

$= \left| \left[\frac{1}{4}x^4\right]_{-1}^{0} \right| + \left[\frac{1}{4}x^4\right]_{0}^{1}$

$= \left|\frac{-1}{4}\right| + \frac{1}{4} = \frac{1}{2}$

14 (a)

Solution :

$\because f''(x) = 6x - 4 \qquad \therefore f'(x) = 3x^2 - 4x + c_1$

At $x = 1$, then $f'(x) = 0$

$\therefore 3 \times 1 - 4 \times 1 + c_1 = 0 \qquad \therefore c_1 = 1$

$\therefore f'(x) = 3x^2 - 4x + 1$

$\therefore f(x) = x^3 - 2x^2 + x + c_2$

, $\because$ the curve passes through (1 , 5) :

$\therefore 5 = 1 - 2 + 1 + c_2 \qquad \therefore c_2 = 5$

$\therefore f(x) = x^3 - 2x^2 + x + 5$

---

15 (d)

Solution :

$f(x) = (x - 2)e^x$

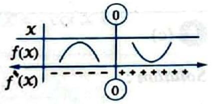
$\therefore f'(x) = e^x + (x - 2)e^x = e^x(x - 1)$

$\therefore f''(x) = e^x + e^x(x - 1) = x e^x$

Put $f''(x)$ = zero

$\therefore e^x = 0$ (refused) or $x = 0$

$\therefore$ The curve is convex downward in $]0 , \infty[$



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16 (d)

Solution :

Let the length of two sides of the triangle are L , M and measure of the included angle is θ

$\therefore A = \frac{1}{2} LM \sin\theta$

$\therefore \frac{dA}{dt} = \frac{1}{2} M \sin\theta \frac{dL}{dt} + \frac{1}{2} L \sin\theta \frac{dM}{dt}$

$+ \frac{1}{2} LM \cos\theta \frac{d\theta}{dt}$

at L = M = 10 , $\theta = \frac{\pi}{3}$, then :

$\frac{dA}{dt} = \frac{1}{2} \times 10 \times \frac{\sqrt{3}}{2} \times 0.1 + \frac{1}{2} \times 10$

$\times \frac{\sqrt{3}}{2} \times 0.1 + \frac{1}{2} \times 10 \times 10 \times \frac{1}{2} \times \frac{1}{5}$

$\simeq 5.866$ cm²/sec.

17 (b)

Solution :

$\frac{dy}{dx} = \cos x + \sec x \tan x \qquad , \frac{dx}{dz} = 3\pi$

$\therefore \frac{dy}{dz} = \frac{dy}{dx} \times \frac{dx}{dz} = 3\pi(\cos x + \sec x \tan x)$

at z = 1 , then $x = 3\pi$

$\left[\frac{dy}{dz}\right]_{x = 3\pi} = 3\pi(\cos 3\pi + \sec 3\pi \tan 3\pi)$

$= 3\pi(-1 + -1 \times 0) = -3\pi$

---

18 (c)

Solution :

$f(x) = 3x - 2$

$(f \circ f)(x) = 3(3x - 2) - 2 = 9x - 8$

$(f \circ f)'(x) = 9$

$\therefore (f \circ f)'(1) = 9$

---

19 (c)

Solution :

$f(x) = |x - 1| \begin{cases} -x + 1 & , x < 1 \\ x - 1 & , x \geq 1 \end{cases}$

$\therefore \int_{-1}^{3} f(x) \, dx = \int_{-1}^{1} (-x + 1) \, dx$

$+ \int_{1}^{3} (x - 1) \, dx$

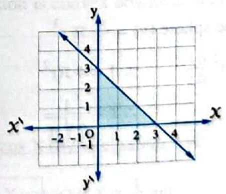
$= \left[\frac{-1}{2}x^2 + x\right]_{-1}^{1} + \left[\frac{1}{2}x^2 - x\right]_{1}^{3}$

$= 2 + 2 = 4$

---

20 (a)

Solution :



The volume $= \pi \int_{0}^{3} y^2 \, dx = \pi \int_{0}^{3} (3 - x)^2 \, dx$

$= \pi \left[\frac{-1}{3}(3 - x)^3\right]_{0}^{3} = 9\pi$ cubic unit.

21 (b)

Solution :

$y = \ln(\tan x)$

$\therefore \frac{dy}{dx} = \frac{\sec^2 x}{\tan x} = \frac{1}{\cos^2 x} \times \frac{\cos x}{\sin x}$

$= \frac{1}{\sin x \cos x} = \frac{1}{\frac{1}{2}\sin 2x} = 2 \csc 2x$

---

22 (d)

Solution :

$f'(x) = g'(x) - 3h'(x)$

$\therefore f'(5) = g'(5) - 3h'(5) = 1 - 3 \times -1 = 4$

---

23 (d)

Solution :

$\int \cos^{99} x \times \sec^{100} x \, dx$

$= \int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx$

$= \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$

(Note that nomerator is the derivative of the denominator)

$= \ln|\sec x + \tan x| + c$

---

24 (d)

Solution :

$f(x) = ax^3 + bx^2$, the point (1 , 12) on it

$\therefore f(1) = 12 \qquad \therefore a + b = 12 \qquad (1)$

$, f'(x) = 3ax^2 + 2bx$

$f''(x) = 6ax + 2b$

at $x = 1 \qquad \therefore f''(x) = 0$

$\therefore$ zero $= 6a + 2b \qquad (2)$

by solving the two equation (1) , (2) :

$\therefore 2a + b = 6$

---

25 (d)

Solution :

$y = -\sin x \qquad \therefore \frac{dy}{dx} = -\cos x$

$\therefore \frac{d^2y}{dx^2} = \sin x$

$\therefore \frac{d^2y}{dx^2} + y = \sin x + (-\sin x) = 0$

Exam 8

1 (a)

Solution :

$$\lim_{x\to 0}\frac{e^x-\sin x-1}{x}=\lim_{x\to 0}\left(\frac{e^x-1}{x}-\frac{\sin x}{x}\right)$$

$$=1-1=\text{zero}$$

---

2 (b)

---

3 (d)

Solution :

$\because$ The tangent is perpendicular to x-axis

$\therefore \left(\frac{dy}{dx}\right)$ is undefined

i.e. $\frac{dx}{dy}=\text{zero}$

---

4 (d)

Solution :

$x=a(\cos\theta+\theta\sin\theta)$

$\therefore \frac{dx}{d\theta}=a(-\sin\theta+\theta\cos\theta+\sin\theta)=a\,\theta\cos\theta$

, $y=a(\sin\theta-\theta\cos\theta)$

$\therefore \frac{dy}{d\theta}=a(\cos\theta+\theta\sin\theta-\cos\theta)=a\,\theta\sin\theta$

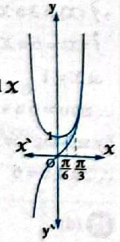
$\therefore \frac{dy}{dx}=\frac{a\,\theta\sin\theta}{a\,\theta\cos\theta}=\tan\theta$

---

5 (a)

Solution :

$\because \sec x>\tan x$ in $\left[\frac{\pi}{6},\frac{\pi}{3}\right]$



$\therefore$ The volume $=\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sec^2 x-\tan^2 x\,dx$

$=\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(\sec^2 x-\sec^2 x+1)\,dx$

$=\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}1\,dx=\pi\,[x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$=\pi\left[\frac{\pi}{3}-\frac{\pi}{6}\right]=\frac{1}{6}\pi^2$

---

6 (b)

Solution :

$\because f(x)=x-\ln x$ $\qquad \therefore f'(x)=1-\frac{1}{x}$

, put $f'(x)=0$ $\qquad \therefore x=1\in\left[\frac{1}{e},e\right]$

, $\because f\left(\frac{1}{e}\right)=\frac{1}{e}-\ln\frac{1}{e}=\frac{1}{e}+1\approx 1.37$

, $f(1)=1-\ln 1=1$

, $f(e)=e-\ln e=e-1\approx 1.7$

$\therefore$ The function has absolute maximum value $=e-1$

---

7 (c)

Solution :

Put $y=\sqrt{x^2+16}$, $z=\frac{x}{x-1}$

$\therefore \frac{dy}{dx}=\frac{2x}{2\sqrt{x^2+16}}=\frac{x}{\sqrt{x^2+16}}$

, $\frac{dz}{dx}=\frac{(x-1)-x}{(x-1)^2}=\frac{-1}{(x-1)^2}$

$\therefore \frac{dy}{dz}=\frac{dy}{dx}\div\frac{dz}{dx}=\frac{x}{\sqrt{x^2+16}}\times\left(-(x-1)^2\right)$

$\therefore \left(\frac{dy}{dz}\right)_{x=3}=\frac{3}{5}\times -4=\frac{-12}{5}$

---

8 (c)

Solution :

$\int\frac{\ln x^2}{\ln x}\,dx=\int\frac{2\ln x}{\ln x}\,dx=\int 2\,dx=2x+c$

---

9 (b)

Solution :

$\int_{-1}^{2} f'(x)\,dx=[2x^2+x-3]_{-1}^{2}=7-(-2)=9$

---

10 (d)

Solution :

volume of the sphere (v) $=\frac{4}{3}\pi r^3$

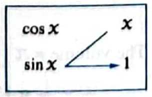
$\therefore \frac{dv}{dt}=4\pi r^2\frac{dr}{dt}$ $\qquad \therefore 1=4\pi r^2$

$\therefore r^2=\frac{1}{4\pi}$ $\qquad \therefore r=\frac{1}{2\sqrt{\pi}}$

---

11 (d)

Solution :



$\therefore \int x\cos x\,dx$

$=x\sin x-\int\sin x\,dx=x\sin x+\cos x+c$

---

12 (d)

Solution :

$y=(2x-c)^3+4$

$\therefore y'=6(2x-c)^2$

$\therefore y''=24(2x-c)$

$\because$ The curve has an inflection point at $x=5$

$\therefore y''=0$ at $x=5$

$\therefore 24(10-c)=0$ $\qquad \therefore c=10$

---

13 (d)

Solution :

$y=e^x$ $\qquad \therefore \frac{dy}{dx}=e^x$

, $z=\sin x$ $\qquad \therefore \frac{dz}{dx}=\cos x$

$\therefore \frac{dy}{dz}=\frac{e^x}{\cos x}$

---

14 (a)

Solution :

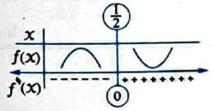
$f(x)=2x^3-3x^2-36x+14$

$\therefore f'(x)=6x^2-6x-36$

$\therefore f''(x)=12x-6$

, put $f''(x)=0$ $\qquad \therefore 12x-6=0$

$\therefore x=\frac{1}{2}$



$\therefore$ The function is convex downwards in $\left]\frac{1}{2},\infty\right[$

---

15 (c)

Solution :

$\frac{dy}{dx}=\sin x\cos x$ (by integration with respect to x)

$\therefore y=\int(\sin x\cos x)\,dx=\frac{1}{2}\sin^2 x+c$

At $x=\frac{\pi}{6}$, $y=1$ $\qquad \therefore 1=\frac{1}{2}\sin^2\frac{\pi}{6}+c$

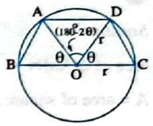
$\therefore c=\frac{7}{8}$ $\qquad \therefore y=\frac{1}{2}\sin^2 x+\frac{7}{8}$

---

16 (b)

Solution :

Area of trapezium A = area of ΔOBA + area of ΔOAD + area of ΔODC



$=\frac{1}{2}r^2\sin\theta+\frac{1}{2}r^2\sin(180^\circ-2\theta)+\frac{1}{2}r^2\sin\theta$

$=r^2\sin\theta+\frac{1}{2}r^2\sin 2\theta$

$\therefore A'=r^2\cos\theta+\frac{1}{2}r^2\cos 2\theta\times 2$

$=r^2\cos\theta+(2\cos^2\theta-1)r^2$

, put $A'=0$

$\therefore r^2(2\cos^2\theta+\cos\theta-1)=0$

$\therefore \cos\theta=\frac{1}{2}$ or $\cos\theta=-1$ (refused).

$\therefore \theta=60^\circ$

, $\because A''=r^2(-4\cos\theta\sin\theta-\sin\theta)$

$\therefore (A'')_{\theta=60^\circ}<0$

$\therefore$ at $=\theta\ 60^\circ$ there is maximum value

Then the base angle is of measure $=\frac{180^\circ-60^\circ}{2}=60^\circ$

---

17 (d)

---

18 (b)

Solution :

$y=a\,x^n-b\,x^{n-1}$

$\therefore \frac{d^ny}{dx^n}=a\,\lfloor n$ $\qquad$ (constant function)

and this function is represented by straight line , parallel to x-axis.

---

19 (c)

Solution :

$\int_0^4 f(x)\,dx=$ area of the rectangle

$-$ are of the shaded surface.

$=24-9=15$

---

20 (b)

Solution :

$\int_{2020}^{2022}(x-2021)^2\,dx=\left[\frac{1}{3}(x-2021)^3\right]_{2020}^{2022}$

$=\frac{1}{3}\left[1^3-(-1)^3\right]=\frac{2}{3}$

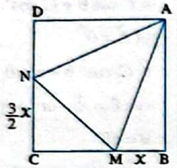
21 (b)

Solution :

$$\int \frac{\sin^{10} x}{\cos^{12} x} dx = \int \tan^{10} x \sec^2 x \, dx$$
$$= \frac{1}{11} \tan^{11} x + c$$

22 (a)

Solution :



area of Δ AMN

A = area of square ABCD

$-[$A. of Δ (ABM)

$+$ A. of Δ (MCN)

$+$ A. of Δ (AND)$]$

$$= 100 - \left[\frac{1}{2} \times x \times 10 + \frac{1}{2}(10 - x) \times \frac{3}{2} x + \frac{1}{2}\left(10 - \frac{3}{2} x\right) \times 10\right] = \frac{3}{4} x^2 - 5x + 50$$

$\therefore \acute{A} = \frac{3}{2} x - 5$, at $\acute{A} = 0$, then $x = \frac{10}{3}$

, $\grave{A} = \frac{3}{2} > 0$

$\therefore$ At $x = \frac{10}{3}$ the area is minimum.

23 (d)

Solution :

$$\int \frac{2x^3}{x^4+5} dx = \frac{1}{2} \int \frac{4x^3}{x^4+5} dx$$
$$= \frac{1}{2} \ln |x^4 + 5| + c$$

24 (a)

Solution :

$x^2 - xy + y^2 = 27$

(by differentiating both sides with respect to x)

$\therefore 2x - x \frac{dy}{dx} - y + 2y \frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx}(x - 2y) = 2x - y$

$\therefore \frac{dy}{dx} = \frac{2x - y}{x - 2y}$

$\therefore \left(\frac{dy}{dx}\right)_{(6, 3)}$ is undefined.

$\therefore$ The tangent makes an angle of measure 90° with the positive direction of x-axis

25 (b)

Exam 9

1 (c)

Solution :

$x + 4y = 5$

(by differentiating both sides with respect to x)

$\therefore 1 + 4 \frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{-1}{4}$ (slope of the normal)

$\therefore \acute{f}(1) = 4$

2 (b)

Solution :

$y = \ln(\sec x + \tan x)$

$$\therefore \frac{dy}{dx} = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$
$$= \frac{\sec x (\tan x + \sec x)}{\sec x + \tan x} = \sec x$$

3 (d)

4 (b)

Solution :

$$\int_{-2}^{2} (a x^3 + b x + c) \, dx$$
$$= \left[\frac{1}{4} a x^4 + \frac{1}{2} b x^2 + c x\right]_{-2}^{2}$$
$$= (4a + 2b + 2c) - (4a + 2b - 2c) = 4c$$

$\therefore$ Then it depends on value of c

5 (a)

Solution :

$$\lim_{x \to 0} \frac{\ln(1 + a x)}{b x} = \frac{a}{b}$$

$\therefore \frac{a}{b} = -1$ $\quad \therefore a = -b$ $\quad \therefore a + b = $ zero

6 (a)

Solution :

$\acute{f}(x) = x . f(x)$

$\therefore \grave{f}(x) = x . \acute{f}(x) + f(x) = x . (x f(x)) + f(x)$

$= x^2 . f(x) + f(x)$

$\therefore \grave{f}(3) = 9 \times (-5) + (-5) = -50$

7 (c)

Solution :

$f(x) = x e^{-x}$

$\therefore \acute{f}(x) = -x e^{-x} + e^{-x} = e^{-x}(1 - x)$

put : $\acute{f}(x) = 0 \quad \therefore e^{-x}(1 - x) = 0$

Then $x = 1 \in [0, 2]$

$\because f(0) = $ zero, $f(1) = \frac{1}{e}$, $f(2) = \frac{2}{e^2}$

$\therefore$ The function has absolute minimum value $=$ zero at $x =$ zero

8 (b)

Solution :

$\because x^2 y^3 = 108$

$\therefore 2x \frac{dx}{dt} y^3 + x^2 \times 3y^2 \frac{dy}{dt} = $ zero

put $x = 2$, $y = 3$, $\frac{dx}{dt} = 2$

$\therefore 216 + 108 \frac{dy}{dt} = $ zero

$\therefore \frac{dy}{dt} = -2$

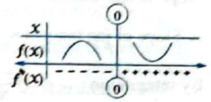
9 (b)

Solution :

$$\int e^{x^3 + 2\ln x} dx = \int (e^{x^3} \times e^{2 \ln x}) \, dx$$
$$= \int e^{x^3} \times x^2 \, dx$$
$$= \frac{1}{3} \int 3x^2 e^{x^3} dx$$
$$= \frac{1}{3} e^{x^3} + c$$

10 (a)

Solution :



$f(x) = x^3 - 12x$

$\therefore \acute{f}(x) = 3x^2 - 12$

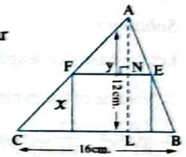
$, \grave{f}(x) = 6x$

$\therefore$ The curve is convex upwards in $]-\infty, 0[$

11 (c)

Solution :

Let dimensions of rectangle be x cm. and y cm. as in the figure.



$\because \Delta\Delta$ AEF, ABC are similar

$\therefore$ The corresponding hieght and corresponding side lengths are proportional

$\therefore \frac{y}{16} = \frac{AN}{AL} = \frac{12 - x}{12}$

$\therefore y = 16 - \frac{4}{3} x$

, Area of the rectangle $A = x y = 16x - \frac{4}{3} x^2$

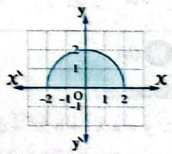
$\therefore \acute{A} = 16 - \frac{8}{3} x$, $\acute{A} = $ zero at $x = 6$

$, \grave{A} = -\frac{8}{3} < 0$ (Maximum)

$\therefore$ Dimensions of the rectangle are 6 and 8 cm.

12 (c)

Solution :



$y = \sqrt{4 - x^2}$

$\therefore y^2 = 4 - x^2$

$\therefore x^2 + y^2 = 4$

it is an equation of a circle

, its centre is the origin point

and its radius is 2 units

$\therefore y = \sqrt{4 - x^2}$ is a part of circle equation that lies above x-axis

$\therefore$ The required area $= \frac{1}{2} \pi (2)^2 = 2\pi$ square units.

13 (c)

Solution :

Slope of the normal $= 3 - x$

$\therefore$ Slope of the tangent $= \frac{-1}{3-x} = \frac{1}{x-3}$

By integration : $\therefore y = \int \frac{1}{x-3}\, dx = \ln|x-3| + c$

$\because A \in$ the curve.

$\therefore \ln|2-3| + c = 3$

$\therefore c = 3$

$\therefore$ The equation is $y = \ln|x-3| + 3$

14 (a)

Solution :

Let the base side length = length of hieght $= x$

$\therefore$ Volume of the pyramid $= \frac{1}{3} x^3 \text{ cm}^3$

$\therefore \frac{dv}{dt} = x^2 \frac{dx}{dt}$

$\therefore 1 = x^2 \times 0.01$

$\therefore x^2 = 100 \qquad \therefore x = 10$ cm.

$\therefore$ Length of base side length = 10 cm.

15 (a)

Solution :

$\because x = \frac{2\tan y}{1-\tan^2 y} = \tan 2y$

(Differentiate with respect to x)

$\therefore 1 = 2\sec^2 2y \cdot \frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{1}{2\sec^2 2y} = \frac{1}{2}\cos^2 2y$

16 (b)

Solution :

$\because y = a x^3 + b x^2 + c x + d$

$\therefore \acute{y} = 3 a x^2 + 2 b x + c$

$\because$ The function has a critical point at (1 , 5)

$\therefore y = 5$ at $x = 1 \qquad \therefore a + b + c + d = 5 \qquad (1)$

, $\acute{y} = 0$ at $x = 1 \qquad \therefore 3a + 2b + c = 0 \qquad (2)$

Subtract (1) from (2) : $\therefore 2a + b - d = -5$

17 (b)

Solution :

$\because x = \frac{t}{t+1}, \; y = \frac{t+1}{t} \qquad \therefore y = \frac{1}{x}$

$\therefore \frac{dy}{dx} = \frac{-1}{x^2} \qquad \therefore \frac{d^2y}{dx^2} = \frac{2}{x^3}$

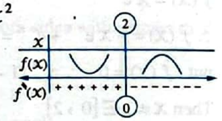
18 (a)

Solution :

$\because f(x) = 15x + 6x^2 - x^3$

$\therefore \acute{f}(x) = 15 + 12x - 3x^2$

$\therefore \ddot{f}(x) = 12 - 6x$



Put $\ddot{f}(x) = 0 : \therefore x = 2$

, $f(2) = 46$

$\therefore$ There is an inflection point at (2 , 46)

19 (a)

20 (b)

Solution :

$\int_{-5}^{8} f(x)\, dx = \int_{-5}^{4} f(x)\, dx + \int_{4}^{8} f(x)\, dx$

$= 19 + (-7) = 12$

21 (a)

Solution :

$\because y_1 = x^2, \; y_2 = -x^2 + 2x$

put $y_1 = y_2 \qquad \therefore x = 1$ or $x = 0$

$\therefore$ The volume $= \pi \int_0^1 (y_2^2 - y_1^2)\, dx$

$= \pi \int_0^1 [(-x^2+2x)^2 - (x^2)^2]\, dx$

$= \pi \int_0^1 (x^4 - 4x^3 + 4x^2 - x^4)\, dx$

$= \pi \int_0^1 (-4x^3 + 4x^2)\, dx$

$= \pi \left[-x^4 + \frac{4}{3}x^3\right]_0^1$

$= \pi\left(\frac{1}{3} - 0\right) = \frac{1}{3}\pi$

22 (a)

Solution :

$\because$ Equation of the tangent is $y = 4x - 2$

$\therefore$ slope of tangent = 4

$\because \acute{f}(x) = 3ax^2 + \frac{1}{\sqrt{x}}$

$\therefore \acute{f}(1) = 4$

$\therefore 3a + 1 = 4$

$\therefore a = 1$

23 (d)

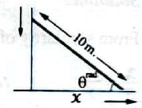
Solution :

Let the inclined angle to the horizontal

$= \theta$ (in radian)

$\because \cos\theta = \frac{x}{10}$

$\therefore x = 10\cos\theta$



$\therefore \frac{dx}{dt} = -10\sin\theta \frac{d\theta}{dt}$

$\therefore$ At $x = 8$, then $\sin\theta = \frac{6}{10} = \frac{3}{5}$

$\therefore 2 = -10 \times \frac{3}{5} \times \frac{d\theta}{dt}$

$\therefore \frac{d\theta}{dt} = -\frac{1}{3}$ rad./min.

24 (d)

Solution :

$\int \tan x\, dx = -\int \frac{-\sin x}{\cos x}\, dx$

$= -\ln|\cos x| + c$

$= \ln|\sec x| + c$

25 (c)

Solution :

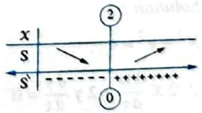
Let the length of $\overline{OA} = s$

, the point on the curve $\left(x, \frac{4}{x}\right)$

$\therefore s^2 = (x-0)^2 + \left(\frac{4}{x} - 0\right)^2$

$= x^2 + \frac{16}{x^2}$

$\therefore 2s\frac{ds}{dx} = 2x - \frac{32}{x^3}$



Put $\frac{ds}{dx} = 0 : \therefore 2x - \frac{32}{x^3} = 0$

$\therefore x^4 = 16$

$\therefore x = 2$ (The negative answer is rejected)

$\therefore$ The least length of the line segment $\overline{OA}$ at $x = 2$

is : $[s]_{x=2} = \sqrt{4 + \frac{16}{4}} = 2\sqrt{2}$ length unit.

Exam 10

1 (b)

Solution :

The slope of the straight line

$y + x - 1 = 0$ equals -1

, $\acute{f}(x) = 2x - 3 \qquad \therefore 2x - 3 = -1$

$\therefore x = 1$

, from the equation of the straight line $\qquad \therefore y = 0$

$\therefore$ The point (1 , 0) $\in$ the curve of the function

$\therefore 0 = 1 - 3 + a \qquad \therefore a = 2$

2 (c)

Solution :

$\int_{-2}^{5} f(x)\, dx = \int_{-2}^{3} f(x)\, dx + \int_{3}^{5} f(x)\, dx$

$\therefore 16 = 12 + \int_3^5 f(x)\, dx$

$\therefore \int_3^5 f(x)\, dx = 4$

3 (b)

Solution :

$f(x) = x^4 - 2x^2$

$\therefore \acute{f}(x) = 4x^3 - 4x = 4x(x^2 - 1)$

$= 4x(x-1)(x+1)$

put : $\acute{f}(x) = 0 \quad \therefore x = 0$ or $x = 1$ or $x = -1$

$\because \ddot{f}(x) = 12x^2 - 4$

$\because \ddot{f}(0) < 0 \qquad \therefore$ At $x = 0$

There exist a local maximum value $= f(0) =$ zero

$\because \ddot{f}(1) > 0 \qquad \therefore$ At $x = 1$

There exist local minimum value

$= f(1) = 1 - 2 = -1$

$\because f'(-1) > 0 \quad \therefore$ At $x = -1$

There exist a local minimum value $= f(-1) = 1 - 2$

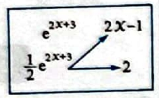
$= -1$

4 (b)

Solution :

$\therefore \int z\, dy = \int e^{2x+3}\, dx$

$= \frac{1}{2} e^{2x+3} + c$



5 (b)

Solution :

$\int \frac{x^3}{x^4+3}\, dx = \frac{1}{4} \int \frac{4x^3}{x^4+3}\, dx = \frac{1}{4} \ln|x^4+3| + c$

6 (b)

Solution :

$f'(x) = \frac{x(2x) - (x^2+9)}{x^2} = \frac{2x^2 - x^2 - 9}{x^2} = \frac{x^2-9}{x^2}$

$\therefore f'(x) = 0$ at $x^2 - 9 = 0$

$\therefore x = 3$ or $x = -3$, $f'(x)$ is undefined at $x = 0$

, to find the absolute maximum and absolute minimum in the interval $[1, 6]$

• $f(1) = \frac{1+9}{1} = 10$ • $f(3) = \frac{9+9}{3} = 6$

• $f(6) = \frac{36+9}{6} = 7.5$

$\therefore$ absolute minimum in the interval $[1, 6]$ is 6

7 (b)

Solution :

$\int \sec^5 x \tan x\, dx = \int \sec^4 x (\sec x . \tan x)\, dx$

$= \frac{1}{5} \sec^5 x + c$

8 (d)

Solution :

$y = \sin^2 x$

$\therefore \frac{dy}{dx} = 2 \sin x \cos x = \sin 2x$

$\therefore \frac{d^2y}{dx^2} = 2 \cos 2x$

$\therefore \frac{d^3y}{dx^3} = -4 \sin 2x$

9 (c)

Solution :

Volume $= \pi \int_1^4 x^2\, dy = \pi \int_1^4 \frac{1}{y}\, dy$

$= \pi [\ln|y|]_1^4$

$= \pi [\ln 4 - \text{zero}]$

$= \pi \ln 2^2 = 2\pi \ln 2$

10 (a)

Solution :

From similarity of the two shaded triangles

$\frac{x}{3} = \frac{2}{y} \qquad \therefore y = \frac{6}{x}$

The area of the triangle AOB is

$A = \frac{1}{2}(x+3)(y+2) = \frac{1}{2}(x+3)\left(\frac{6}{x}+2\right)$

$= 6 + x + 9x^{-1}$

$\therefore A' = 1 - 9x^{-2}$

$\therefore A'' = 18x^{-3}$

Put $A' = 0$

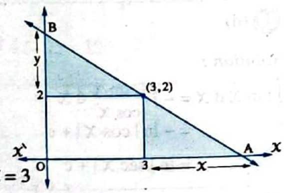
, then $1 - \frac{9}{x^2} = 0$

$\therefore x^2 = 9$, then $x = 3$

, $\because A''_{\text{at } x=3} > 0$ (minimum value)

$\therefore$ smallest area $= A_{(\text{at } x=3)} = 6 + 3 + \frac{9}{3}$

$= 12$ square units.



11 (c)

Solution :

$x^2 + y^2 = 4$

$\therefore 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$

when the two ends move by the same speed

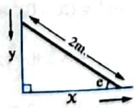
, then $\frac{dy}{dt} = -\frac{dx}{dt}$

$\therefore 2x \frac{dx}{dt} - 2y \frac{dx}{dt} = 0$.

$\therefore x - y = 0 \quad$ *i.e.* $x = y$

$\therefore x^2 + y^2 = 4 \quad \therefore 2x^2 = 4 \qquad \therefore x = \sqrt{2}$

i.e. The lower end distance $\sqrt{2}$ m. from the vertical wall



12 (b)

Solution :

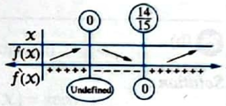
$f(x) = \sqrt[3]{x^2}\,(3x - 7)$

$\therefore f'(x) = 3x^{\frac{2}{3}} + (3x - 7) \times \frac{2}{3} x^{-\frac{1}{3}}$

$= \frac{9x + 6x - 14}{3x^{\frac{1}{3}}}$

Put $f'(x) = 0$

$\therefore x = \frac{14}{15}$



, $f'(x)$ undefined at $x = 0$

$\therefore$ The function is increasing at

$]-\infty, 0[\ , \ ]\frac{14}{15}, \infty[$

$\therefore a = 0 , b = \frac{14}{15} \qquad \therefore 7a + 30b = 28$

13 (a)

Solution :

$x = \cos\theta \qquad \therefore \frac{dx}{d\theta} = -\sin\theta$

$, y = \sqrt{2} + \sin\theta \qquad \therefore \frac{dy}{d\theta} = \cos\theta$

$\therefore \frac{dy}{dx} = \cos\theta \div (-\sin\theta) = -\cot\theta$

$\therefore \left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{4}} = -1$

$\therefore$ Slope of the normal to the curve $= 1$

14 (d)

15 (b)

Solution :

$f(x) = \frac{x^{65}}{\lfloor 65}$

$\therefore f'(x) = \frac{64\, x^{64}}{\lfloor 65}$

, $f''(x) = \frac{65 \times 64\, x^{63}}{\lfloor 65}$, etc ...

then $f^{(65)}(x) = \frac{\lfloor 65\,(1)}{\lfloor 65} = 1$

16 (d)

Solution :

$\because A(3, 5) \in$ curve $\qquad \therefore f(3) = 5$

$\because$ Tangent makes an angle of measure 45° with positive direction of x-axis

$\therefore$ Then its slope $= 1 \qquad \therefore f'(3) = 1$

$\because g(x) = x f(x) \qquad \therefore g'(x) = f(x) + x f'(x)$

$\therefore g'(3) = f(3) + 3 \times f'(3) = 5 + 3 \times 1 = 8$

17 (d)

Solution :

$\because h = 2r$

$\therefore$ volume of the cone "v" $= \frac{1}{3} \pi r^2 h$

$\therefore v = \frac{2}{3} \pi r^3$

$\therefore \frac{dv}{dt} = 2\pi r^2 \frac{dr}{dt} \qquad \because \frac{dr}{dt} = \frac{1}{\pi}$ cm./sec.

$\therefore \left(\frac{dv}{dt}\right)_{r=5} = 2\pi (5)^2 \times \frac{1}{\pi} = 50$ cm.^{3} / sec.

18 (c)

Solution :

$\lim_{x \to 0} \frac{6^x - 1}{3x} = \frac{1}{3} \lim_{x \to 0} \frac{6^x - 1}{x} = \frac{1}{3} \ln 6$

19 (c)

Solution :

$\because y^x = x \qquad (1)$

(by taking logarithm for both sides to the base e)

$\therefore \ln y^x = \ln x$

$\therefore x \ln y = \ln x$ (by differentiating both sides respect to x)

$\therefore \ln y + x \times \frac{1}{y} \times y' = \frac{1}{x}$

when $\frac{dy}{dx} = 0$

$\therefore \ln y + \text{zero} = \frac{1}{x}$

$\therefore x \ln y = 1$ $\therefore \ln y^{x} = 1$, from (1)

$\therefore \ln x = 1$ $\therefore x = e$

20 (b)

Solution :

(by multiplying both numerator and denominator by e^x)

$\therefore \int \frac{dx}{e^x + e^{-x} + 2} = \int \frac{e^x}{e^{2x} + 1 + 2e^x} dx$

$= \int \frac{e^x}{(e^x+1)^2} dx = -(e^x + 1)^{-1} + c$

$= -\frac{1}{e^x + 1} + c$

21 (d)

Solution :

$\because f(x) = x^4 - 4x^2$, its domain is $\mathbb{R}$

$\therefore f'(x) = 4x^3 - 8x$

, put $f'(x) = 0$ $\therefore 4x(x^2 - 2) = 0$

$\therefore x = 0$ or $x = \sqrt{2}$ or $x = -\sqrt{2}$

, $\because f''(x) = 12x^2 - 8$

$\therefore f''(0) < 0$, $f''(\sqrt{2}) > 0$, $f''(-\sqrt{2}) > 0$

$\therefore f(0) = 0$ (local maximum)

, $f(\sqrt{2}) = f(-\sqrt{2}) = -4$ (local minimum)

22 (d)

Solution :

The coordinates of B $(x , y) = (x , x^2 - 5x + 14)$

The perimeter of the rectangle p

$= 2(x + y) = 2(x + x^2 - 5x + 14)$

$= 2(x^2 - 4x + 14)$

$\therefore \frac{dp}{dx} = 4x - 8$

Put $\frac{dp}{dx} = 0 : \therefore x = 2$

$\because \frac{d^2p}{dx^2} = 4 > 0$

$\therefore$ The perimeter of the rectangle is minimum at $x = 2$

$[p]_{x=2} = 20$ length units.

23 (c)

Solution :

$\int_{-1}^{4} f(x)\,dx = \int_{-1}^{2} (2x - 1)\,dx + \int_{2}^{4} 3\,dx$

$= [x^2 - x]_{-1}^{2} + [3x]_{2}^{4}$

$= [2 - 2] + 3[4 - 2] = 6$

24 (c)

Solution :

$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx$

$= (2 \times 5 - A_1) + A_2$

$= 10 - 2 + 7 = 15$ square units

25 (b)

Solution :

$\int (\sin^2 x + \sin^2 x \tan^2 x)\,dx = \int \sin^2 x (1 + \tan^2 x)\,dx$

$= \int \sin^2 x \cdot \sec^2 x\,dx$

$= \int \tan^2 x\,dx$

$= \int (\sec^2 x - 1)\,dx$

$= \tan x - x + c$

Exam 11

1 (b)

Solution :

$f(x) = 2x^3$

$\therefore f'(x) = 6x^2$

$\therefore$ The rate of change of slope to the tangent $= 12x$

at $x = 3$

$\therefore$ The rate of change of slope to the tangent

$= 12 \times 3 = 36$

2 (b)

Solution :

$y = x^x$

(by taking logarithm of both sides to the base "e")

$\therefore \ln y = x \ln x$

(by differentiating both sides with respect to x)

$\therefore \frac{y'}{y} = x \times \frac{1}{x} + \ln x$

$\therefore y' = y(1 + \ln x)$

$\therefore y' = x^x (1 + \ln x)$

Put $y' = 0$ $\therefore x^x = 0$ (refused)

or $1 + \ln = \text{zero}$ $\therefore \ln x = -1$

$\therefore x = e^{-1} = \frac{1}{e}$

3 (b)

Solution :

$\therefore \int_{2}^{3} x\sqrt{x^2 + 1}\,dx = \text{constant}$

$\therefore \frac{d}{dx}\left(\int_{2}^{3} x\sqrt{x^2 + 1}\,dx\right) = \text{zero}$

4 (d)

Solution :

Let the point (x , y) lies on the curve $y = \frac{1}{2}x^2 - 4$

$\therefore$ The point is $\left(x , \frac{1}{2}x^2 - 4\right)$

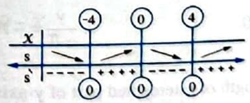
The distance between the two points

$s = \sqrt{(x - 0)^2 + \left(\frac{1}{2}x^2 - 4 - 5\right)^2}$

$= \sqrt{\frac{1}{4}x^4 - 8x^2 + 81}$ (1)

$\therefore s' = \frac{x^3 - 16x}{2\sqrt{\frac{1}{4}x^4 - 8x^2 + 81}}$

Put $s' = 0$ $\therefore x = 0$ or $x = \pm 4$



s has minimum value at $x = -4$, 4

$\therefore$ Nearest points are $(-4 , 4)$ and $(4 , 4)$

, from (1) :

$\therefore s = \sqrt{17}$ length unit.

5 (c)

Solution :

$y = \sin^3 \theta$ $\therefore \frac{dy}{d\theta} = 3\sin^2\theta \cos\theta$

, $\because z = \cos^3\theta$ $\therefore \frac{dz}{d\theta} = -3\cos^2\theta \sin\theta$

$\frac{dy}{dz} = \frac{dy}{d\theta} \div \frac{dz}{d\theta} = \frac{3\sin^2\theta\cos\theta}{-3\cos^2\theta\sin\theta}$

$= \frac{-\sin\theta}{\cos\theta} = -\tan\theta$

6 (d)

Solution :

Slope of the tangent $= \frac{dy}{dx} = x\sqrt{x + 1}$

$\therefore y = \int x\sqrt{x + 1}\,dx$

$= \int (x + 1 - 1)(x + 1)^{\frac{1}{2}}\,dx$

$= \int (x + 1)^{\frac{3}{2}} - (x + 1)^{\frac{1}{2}}\,dx$

$= \frac{2}{5}(x + 1)^{\frac{5}{2}} - \frac{2}{3}(x + 1)^{\frac{3}{2}} + c$

, $\because \left(0 , \frac{11}{15}\right)$ on the curve.

$\therefore \frac{11}{15} = \frac{2}{5}(0 + 1)^{\frac{5}{2}} - \frac{2}{3}(0 + 1)^{\frac{3}{2}} + c$

$\therefore c = 1$

$\therefore$ Equation of the curve

$y = \frac{2}{5}(x + 1)^{\frac{5}{2}} - \frac{2}{3}(x + 1)^{\frac{3}{2}} + 1$

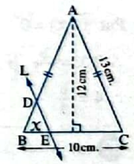
7 (b)

Solution :

$\because \overleftrightarrow{DE} \parallel \overline{AC}$

$\therefore \Delta DEB \sim \Delta ACB$

$\therefore \frac{A(\Delta DEB)}{A(\Delta DEB)} = \left(\frac{BE}{BC}\right)^2$



, let $BE = x$

, $A(\Delta DEB) = A$

$\therefore \frac{A}{\frac{1}{2} \times 10 \times 12} = \frac{x^2}{100}$

$\therefore A = 0.6x^2$ $\therefore \frac{dA}{dt} = 1.2x\frac{dx}{dt}$

, at $x = 1$

$\therefore \frac{dA}{dt} = 1.2 \times 1 \times \frac{1}{10} = 0.12\ cm^2/sec.$

8 (b)

Solution:

$\because y = \sin x(1+\cos x)$ $\quad \therefore y = \sin x + \sin x \cos x$

$\therefore y = \sin x + \frac{1}{2}\sin 2x$ $\quad \therefore \acute{y} = \cos x + \cos 2x$

$, \acute{\acute{y}} = -\sin x - 2\sin 2x$

$\acute{y} = 0 \longrightarrow \cos x + \cos 2x = 0$

$\therefore 2\cos^2 x + \cos x - 1 = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

$2\cos x - 1 = 0 \quad , \quad \cos x = \frac{1}{2}$

$\therefore x = \frac{\pi}{3}$ or $\cos x + 1 = 0$

$\therefore \cos x = -1$ (refused).

$\because \acute{\acute{f}}\left(\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) - 2\sin\left(\frac{2\pi}{3}\right) = \frac{-3\sqrt{3}}{2} < 0$

$\therefore$ At $x = \frac{\pi}{3}$ has local maximum value

$\therefore f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right)$

$= \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{3}{4}\sqrt{3}$

$\therefore$ The local maximum value $= \frac{3}{4}\sqrt{3}$

9 (d)

10 (b)

Solution:

$\int 2\cos^2 x\, dx = \int (1+\cos 2x)\, dx = x + \frac{1}{2}\sin 2x + c$

11 (a)

Solution:

$f(x) = x^2 + \frac{b}{x}$

$\therefore \acute{f}(x) = 2x - b x^{-2} = 2x - \frac{b}{x^2}$

Put $\acute{f}(x) = 0$

$\therefore 2x = \frac{b}{x^2}$, at $(x = 2)$

$\therefore 4 = \frac{b}{4}$ $\quad \therefore b = 16$

12 (a)

Solution:

$\int \frac{1}{1-\sin x} \times \frac{1+\sin x}{1+\sin x}\, dx = \int \frac{1+\sin x}{1-\sin^2 x}\, dx$

$= \int \frac{1+\sin x}{\cos^2 x}\, dx = \int (\sec^2 x + \tan x \sec x)\, dx$

$= \tan x + \sec x + c$

$\therefore a = 1 \; , \; b = 1$ $\quad \therefore a^2 + b^2 = 2$

13 (c)

14 (a)

Solution:

$y = \ln\sqrt{\tan x}$

$\therefore \frac{dy}{dx} = \frac{\sec^2 x}{2\tan x}$ $\quad \therefore \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = \frac{2}{2(1)} = 1$

15 (d)

Solution:

$f(x) = 8x \sin x \cos x \cos 2x$

$= 4x \sin 2x \cos 2x = 2x \sin 4x$

$\therefore \acute{f}(x) = 2\sin 4x + 8x\cos 4x$

$\therefore \acute{f}\left(\frac{\pi}{8}\right) = 2 \times 1 + 8 \times \frac{\pi}{8} \times \text{zero} = 2$

16 (c)

Solution:

$x = (1-y)(1+y+y^2)$

$\therefore x = 1 - y^3$ (by differentiating with respect to x)

$1 = -3y^2 \frac{dy}{dx}$ $\quad \therefore \frac{dy}{dx} = \frac{-1}{3y^2}$

17 (d)

Solution:

$\because y = x \sin x$

$\therefore \frac{dy}{dx} = x\cos x + \sin x$

$\therefore \left(\frac{dy}{dx}\right)_{x=\pi} = \pi \times (-1) + \text{zero} = -\pi$

, at $x = \pi$ $\quad \therefore y = \text{zero}$

$\therefore$ The point is (π , zero)

Equation of the tangent is

$\frac{y}{x-\pi} = -\pi$

, put $x = 0$ $\quad \therefore \frac{y}{-\pi} = -\pi$

$\therefore y = \pi^2$

i.e. The length of intercepted part of y-axis $= \pi^2$

18 (a)

Solution:

The dimensions of the cuboid at any instant t are:

$3 + 2t \; , \; 4 + t \; , \; 12 - 3t$

$\therefore$ The volume of the cuboid at any instant

$V = (3+2t)(4+t)(12-3t)$

$\therefore \frac{dv}{dt} = 2(4+t)(12-3t) + (3+2t)(12-3t)$

$\qquad - 3(3+2t)(4+t)$

$\therefore \left(\frac{dv}{dt}\right)_{t=2} = 2(6)(6) + (7)(6) - 3(7)(6) = -12 \text{ cm}^3/\text{sec.}$

19 (a)

Solution:

$\lim_{x\to 0} \frac{\sin 4x}{3^x - 1} = \lim_{x\to 0} \frac{\frac{\sin 4x}{x}}{\frac{3^x-1}{x}} = \frac{4}{\ln 3}$

20 (d)

Solution:

$f(x) = x + \frac{1}{x}$ $\quad \acute{f}(x) = 1 - \frac{1}{x^2}$

$\therefore \acute{f}(x) = 0$ at $x = 1 \in \left[\frac{1}{2}, 3\right]$

$, x = -1 \notin \left[\frac{1}{2}, 3\right]$

$, \acute{f}(x)$ is undefined at $x = 0 \notin \left[\frac{1}{2}, 3\right]$

$, \because f\left(\frac{1}{2}\right) = 2\frac{1}{2} \; , f(1) = 2 \; , f(3) = 3\frac{1}{3}$

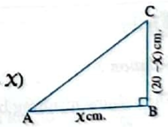
The function has absolute maximum value $= 3\frac{1}{3}$

21 (a)

Solution:

let AB $= x$ cm.

$\therefore$ BC $= (20 - x)$ cm.



$\therefore$ The area (A) $= \frac{1}{2} x \times (20 - x)$

$\therefore A = 10x - \frac{1}{2}x^2$

$\therefore \frac{dA}{dx} = 10 - x$

Putting $\frac{dA}{dx} = 0$

$\therefore 10 - x = 0$ $\quad \therefore x = 10$

$, \because \frac{d^2A}{dx^2} = -1 < 0$

$\therefore x = 10$ makes the area have maximum value.

$\therefore A = 10 \times 10 - \frac{1}{2} \times 100 = 50 \text{ cm}^2$

$\therefore$ The greatest area of the triangle ABC $= 50 \text{ cm}^2$.

22 (a)

Solution:

$\int x(x-5)^3\, dx$

putting $z = x - 5$ $\quad \therefore x = z + 5$

$\therefore dx = dz$

$\therefore \int x(x-5)^3\, dx$

$= \int (z+5)(z)^3\, dz$

$= \int (z^4 + 5z^3)\, dz = \frac{1}{5}z^5 + \frac{5}{4}z^4 + c$

$= \frac{1}{5}(x-5)^5 + \frac{5}{4}(x-5)^4 + c$

23 (a)

Solution:

$\int_0^3 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx$

$= \int_0^2 x^2\,dx + \int_2^3 (3x-2)\,dx$

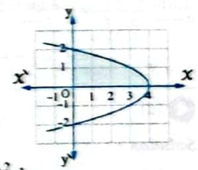
$= \left[\frac{1}{3}x^3\right]_0^2 + \left[\frac{3}{2}x^2 - 2x\right]_2^3 = \frac{8}{3} + \frac{11}{2} = \frac{49}{6}$

24 (a)

Solution:

$y^2 = 4 - x$

About y-axis:



The volume $= \pi \int_0^2 x^2\, dy$

$= \pi \int_0^2 (4 - y^2)^2\, dy$

$= \pi \int_0^2 (16 - 8y^2 + y^4)\, dy$

$= \pi \left[16y - \frac{8}{3}y^3 + \frac{1}{5}y^5\right]_0^2$

$= \frac{256}{15}\pi$ cubic unit.

25 (d)

Solution:

$\int \sin^2 x\, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

Exam 12

1 (b)

Solution :

$y = x^{\frac{1}{3}}$

$\therefore \frac{dy}{dx} = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

$\therefore \left(\frac{dy}{dx}\right)_{x=0}$ undefined.

$\therefore$ The tangent to the curve of the function is parallel to y-axis

---

2 (d)

Solution :

$\int_1^2 f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^4 f(x)\,dx - \int_2^4 f(x)\,dx$

$\therefore \int_1^2 f(x)\,dx = 5 + 2 - 6 = 1$

$\therefore \int_2^1 f(x)\,dx = -\int_1^2 f(x)\,dx = -1$

---

3 (d)

Solution :

$\lim_{x \to 6} \frac{e^x - e^6}{x - 6} = \lim_{x-6 \to 0} \frac{e^6 (e^{x-6} - 1)}{x - 6}$

$= e^6 \times 1 = e^6$

---

4 (d)

Solution :

$y = \frac{\ln x}{x}$

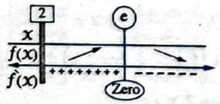
$\therefore \frac{dy}{dx} = \frac{x \times \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$

put $\frac{dy}{dx} = 0$

$\therefore 1 - \ln x = 0 \qquad \therefore \ln x = 1$

$\therefore x = e \in [2, \infty[$

, $\frac{dy}{dx}$ undefined at $x = 0 \notin [2, \infty[$



$\therefore$ The function has local maximum value at $x = e$

, $f(e) = \frac{\ln e}{e} = \frac{1}{e}$

5 (a)

Solution :

$f(x) = \frac{x^2 + 1}{x^2 + 3}$

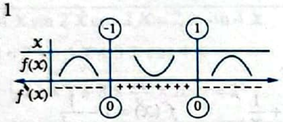
$\therefore f'(x) = \frac{(x^2+3)(2x) - (x^2+1)(2x)}{(x^2+3)^2} = \frac{4x}{(x^2+3)^2}$

, $f''(x) = \frac{(x^2+3)^2(4) - 4x\left(2(x^2+3) \times 2x\right)}{(x^2+3)^4}$

$= \frac{(x^2+3)(4x^2 + 12 - 16x^2)}{(x^2+3)^4} = \frac{-12x^2 + 12}{(x^2+3)^3}$

When $f''(x) = 0$, then $-12x^2 + 12 = 0$

$\therefore x = \pm 1$



The curve is convex downward in $]-1, 1[$

---

6 (a)

Solution :

$x^2 y = 2x + 5$ (by differentiation with respect to x)

$\therefore x^2 \frac{dy}{dx} + 2xy = 2$ (by differentiation with respect to x)

$\therefore x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + 2x \frac{dy}{dx} + 2y = 0$

$\therefore x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = 0$

$\therefore x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} = -2y$

---

7 (b)

Solution :

The volume resulting by the rotate about y-axis

$= \pi \int_0^2 x^2\,dy$

$= \pi \int_0^2 y\,dy$

---

8 (a)

Solution :

Let the side length of the triangle $= x$

$\therefore$ perimeter of the triangle (p) $= 3x$

$\therefore \frac{dp}{dt} = 3\frac{dx}{dt} = 3 \times \frac{1}{3} = 1$ cm./sec.

9 (b)

Solution :

$x = 2t^3 + 3 \qquad \therefore \frac{dx}{dt} = 6t^2$

, $y = t^4 \qquad \therefore \frac{dy}{dt} = 4t^3$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{4t^3}{6t^2} = \frac{2}{3}t$

, $\frac{d^2y}{dx^2} = \frac{2}{3}\frac{dt}{dx} = \frac{2}{3} \times \frac{1}{6t^2} = \frac{1}{9t^2}$

$\therefore \left(\frac{d^2y}{dx^2}\right)_{t=1} = \frac{1}{9(1)^2} = \frac{1}{9}$

---

10 (c)

Solution :

$\int \cot^3 x\,dx = \int \cot^2 x \cot x\,dx$

$= \int (\csc^2 x - 1) \cot x\,dx$

$= \int (\cot x \csc^2 x - \cot x)\,dx$

$= -\frac{1}{2}\cot^2 x + \ln|\csc x| + c$

---

11 (b)

Solution :

$f(x) = x^3 + 4x + 2 \qquad \therefore f'(x) = 3x^2 + 4 > 0$

for all values of $x \in \mathbb{R}$

---

12 (c)

---

13 (a)

Solution :

Let radius of its circle $= x$ and length of the arc $= y$

$\therefore$ Its area $= \frac{1}{2} x y = 16 \qquad \therefore y = \frac{32}{x}$

$\therefore$ The perimeter $P = 2x + y = 2x + \frac{32}{x} = 2x + 32x^{-1}$

$\therefore P' = 2 - 32x^{-2} \qquad , P'' = 64x^{-3} > 0$

, put $P' = 0$

$\therefore x = 4$ (and negative solution is refused)

$\therefore$ P has minimum value at $x = 4$

i.e. Its radius length = 4 cm.

14 (b)

Solution :

Put $y_1 = y_2$

$\therefore 2x^2 = 3 - x^2 \qquad \therefore 3x^2 = 3 \qquad \therefore x = \pm 1$

$\therefore$ Area (A) $= \int_{-1}^1 (y_2 - y_1)\,dx$

$= \int_{-1}^1 (3 - x^2 - 2x^2)\,dx$

$= \int_{-1}^1 (3 - 3x^2)\,dx = [3x - x^3]_{-1}^1$

$= (3 - 1) - \left(3(-1) - (-1)^3\right) = 4$

---

15 (b)

Solution :

Slope of the normal of the curve $= \frac{5 + 2y}{2 - 3x^2}$

$\therefore$ Slope of the tangent $= -\frac{2 - 3x^2}{5 + 2y} = \frac{3x^2 - 2}{5 + 2y}$

$\therefore \frac{dy}{dx} = \frac{3x^2 - 2}{2y + 5}$

$\therefore \int (2y + 5)\,dy = \int (3x^2 - 2)\,dx$

$\therefore y^2 + 5y = x^3 - 2x + c$

$\because$ The curve passes through the point (1 , 2)

$\therefore 4 + 10 = 1 - 2 + c \qquad \therefore c = 15$

$\therefore$ The equation of the curve is : $y^2 + 5y = x^3 - 2x + 15$

---

16 (b)

Solution :

At $x = 1 \qquad \therefore 2 + \ln y \ln 1 = 1^2 + y$

$\therefore y = 1 \qquad \therefore$ the point is (1 , 1)

$\because 2 + \ln y \ln x = x^2 + y$

(by differentiation with respect to x)

$\therefore \ln y \times \frac{1}{x} + \ln x \times \frac{1}{y} \times \frac{dy}{dx} = 2x + \frac{dy}{dx}$

By substitution $x = 1$, $y = 1$

$\therefore \ln(1) \times \frac{1}{1} + \ln 1 \times \frac{1}{1} \times \frac{dy}{dx} = 2(1) + \frac{dy}{dx}$

$\therefore \frac{dy}{dx} = -2$

$\therefore$ Equation of the tangent is : $y - 1 = -2(x - 1)$

$\therefore 2x + y = 3$

17 (b)

Solution :

$\because y = (\sec x - 1)(\sec x + 1)$

$= \sec^2 x - 1 = \tan^2 x$

$\frac{d}{dx}\left((\sec x - 1)(\sec x + 1)\right)$

$= \frac{d}{dx}(\tan^2 x) = 2\tan x . \sec^2 x$

18 (c)

Solution :

$y = a x^3 + b x^2$,

$\because$ (1 , 6) satisfies the equation

$\therefore a + b = 6$ (1)

, $\because \frac{dy}{dx} = 3 a x^2 + 2 b x$

, $\because$ the slope of the given straight line = 13

$\therefore \left(\frac{dy}{dx}\right)_{(1,6)} = 13$

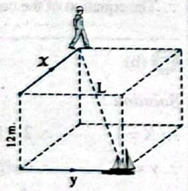
$\therefore 3a + 2b = 13$ (2)

From (1), (2) : $\therefore a = 1$, $b = 5$

$\therefore a b = 5$

19 (a)

Solution :



$\frac{dx}{dt} = 3$ m./min.

, $\frac{dy}{dt} = 6$ m./min.

From the figure :

$L^2 = x^2 + y^2 + (12)^2$ (1)

$\therefore 2L\frac{dL}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

After 6 minutes : $x = 3 \times 6 = 18$ meters.

, $y = 6 \times 6 = 36$ meters

, from (1) : $L = 42$ meters

$\therefore 2 \times 42 \times \frac{dL}{dt} = 2 \times 18 \times 3 + 2 \times 36 \times 6$

$\therefore \frac{dL}{dt} = \frac{45}{7}$ m./min.

20 (d)

Solution :

$x e^{xy} = y + \sin^2 x$ (1)

at $x = 0$ $\therefore 0 = y + 0$ $\therefore y = 0$

(By differentiating (1) with respect to x)

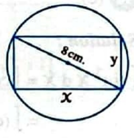
$\therefore e^{xy} + x e^{xy}(y + x\acute{y}) = \acute{y} + 2\sin x \cos x$

, at (0 , 0) $\therefore 1 + 0 = \acute{y} + 0$

$\therefore \acute{y} = 1$ $\therefore \left(\frac{dy}{dx}\right)_{x=0} = 1$

21 (a)

Solution :



$x^2 + y^2 = 64$

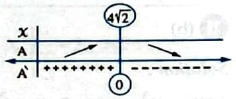
$\therefore y = \sqrt{64 - x^2}$

Area (A) = $x y$

$\therefore A = x\sqrt{64 - x^2}$

$\therefore \acute{A} = x \times \frac{-2x}{2\sqrt{64 - x^2}} + \sqrt{64 - x^2} = \frac{64 - 2x^2}{\sqrt{64 - x^2}}$

putting $\acute{A} = 0$



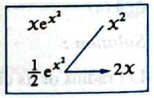
$\therefore x = 4\sqrt{2}$

$\therefore x = 4\sqrt{2}$, $y = 4\sqrt{2}$

$\therefore$ Maximum area $= 4\sqrt{2} \times 4\sqrt{2} = 32$ cm^2

22 (d)

Solution :



$\therefore \int x^3 e^{x^2}\, dx$

$= \int x^2 . x e^{x^2}\, dx$

$= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}\int 2x e^{x^2}\, dx$

$= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + c$

23 (b)

Solution :

$\int_0^6 f(x)\, dx = \int_0^2 2\, dx + \int_2^6 x\, dx$

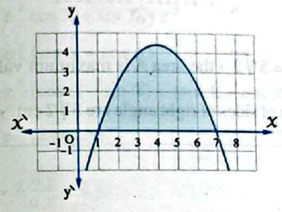
$= [2x]_0^2 + [\frac{1}{2}x^2]_2^6$

$= (4 - 0) + (18 - 2) = 20$

24 (a)

Solution :

The area of hotel entryway



$= \int_1^7 -\frac{1}{2}(x-1)(x-7)\, dx = -\frac{1}{2}\int_1^7 (x^2 - 8x + 7)\, dx$

$= -\frac{1}{2}\left[\frac{1}{3}x^3 - 4x^2 + 7x\right]_1^7 = 18$ square metre.

$\therefore$ The cost of the entryway = 18 × 1500 = L.E. 27000

25 (a)

Solution :

$\because$ g is increasing function $\therefore \acute{g}(x) > 0$

$\because$ k is decreasing function $\therefore \acute{k}(x) < 0$

$\because f(x) = 4g(x) - 3k(x)$

$\therefore \acute{f}(x) = 4\acute{g}(x) - 3\acute{k}(x) > 0$

$\therefore f$ is increasing function

Exam 13

1 (b)

Solution :

$f(x) = x^3 + 4x + 2$ $\therefore \acute{f}(x) = 3x^2 + 4$

$\therefore$ The function is increasing on $\mathbb{R}$

2 (b)

Solution :

$f(x) = e^{2x+1}$ $\therefore \acute{f}(x) = 2e^{2x+1}$

at $\left(\frac{-1}{2}, 1\right)$

$\acute{f}\left(\frac{-1}{2}\right) = 2e^0 = 2$

$\therefore$ Equation of the tangent : $\frac{y-1}{x+\frac{1}{2}} = 2$

$\therefore 2x + 1 = y - 1$

i.e. $y = 2x + 2$

3 (b)

Solution :

$f(x) = x + \frac{1}{x}$

$\therefore \acute{f}(x) = 1 - \frac{1}{x^2}$ $\therefore \acute{f}(x) = 0$, then $x^2 = 1$

$\therefore x = 1$ or $x = -1 \notin \left[\frac{1}{2}, 3\right]$

, $\acute{f}$ is undefined at $x = 0 \notin \left[\frac{1}{2}, 3\right]$

$\because f(1) = 1 + \frac{1}{1} = 2$

$f\left(\frac{1}{2}\right) = \frac{1}{2} + 2 = 2\frac{1}{2}$

$f(3) = 3\frac{1}{3}$

$\therefore$ The function has absolute minimum value = 2

4 (a)

Solution :

Area of regular octagon (A) $= \frac{8}{4}x^2 \cot\frac{180°}{8}$

$\therefore A = 2x^2 \cot\frac{45°}{2}$

$\therefore \frac{dA}{dt} = 4x \cot\frac{45°}{2}\frac{dx}{dt}$

at $x = 10$

$\therefore \frac{dA}{dt} = 4 \times 10 \cot\frac{45°}{2} \times 0.2 = 19.3$ cm^2/sec.

5 (b)

Solution :

$\lim_{x \to 0} \frac{a^x + b^x + c^x - 1 - 1 - 1}{x}$

$= \lim_{x \to 0} \frac{a^x - 1}{x} + \lim_{x \to 0} \frac{b^x - 1}{x} + \lim_{x \to 0} \frac{c^x - 1}{x}$

$= \ln a + \ln b + \ln c = \ln(abc)$

6 (c)

7 (b)

Solution :

The area (A) $= \pi r^2$

$\therefore \frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

$\therefore 20\pi = 2\pi r \times 2$

$\therefore r = 5$ cm.

8 (b)

Solution :

$y = \ln(\sec x + \tan x)$

$\therefore \frac{dy}{dx} = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

$= \frac{\sec x \cancel{(\tan x + \sec x)}}{\cancel{(\tan x + \sec x)}}$

$= \sec x$

9 (c)

Solution :

$\because f$ is one - to - one function

$\therefore f$ is always increasing or always decreasing

$\therefore f'(x) \geq 0$ or $f'(x) \leq 0$

$, \because f'(x) = 3x^2 - 2kx + 12$

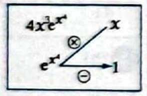
$\because$ discriminant ≤ 0 $\qquad \therefore b^2 - 4ac \leq 0$

$\therefore 4k^2 - 4 \times 3 \times 12 \leq 0$ $\qquad \therefore 4k^2 \leq 144$

$\therefore k^2 \leq 36$ $\qquad \therefore |k| \leq 6$ $\qquad \therefore k \in [-6, 6]$

10 (b)

Solution :



$\int (1 + 4x^4) e^{x^4} dx$

$= \int e^{x^4} dx + \int 4x^4 e^{x^4} dx$

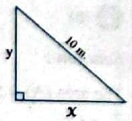
$= \int e^{x^4} dx + \int x \times 4x^3 e^{x^4} dx$

$= \int e^{x^4} dx + x e^{x^4} - \int e^{x^4} dx$

$= x e^{x^4} + c$

11 (b)

Solution :



$x^2 + y^2 = 100$

$\therefore y = \sqrt{100 - x^2}$

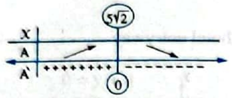
$\therefore$ Area of Δ (A) $= \frac{1}{2} x y$

$\therefore A = \frac{1}{2} x \sqrt{100 - x^2}$

$\therefore A' = \frac{1}{2} x \times \frac{-x}{\sqrt{100 - x^2}} + \frac{1}{2}\sqrt{100 - x^2}$

$= \frac{-x^2 + 100 - x^2}{2\sqrt{100 - x^2}} = \frac{100 - 2x^2}{2\sqrt{100 - x^2}}$

Put $A' = 0$ $\qquad \therefore x = 5\sqrt{2}$



When $x = 5\sqrt{2}$, the area A is maximum value.

$\therefore$ Length of sides of right angle are $5\sqrt{2}$ cm. , $5\sqrt{2}$ cm.

12 (b)

Solution :

$\because f'(x) = 6x^2 - 30x + 36$

$\therefore f'(x) = 6(x^2 - 5x + 6)$

$\therefore f'(x) = 6(x - 2)(x - 3)$

$\because f'(x) = 0$ $\qquad$ When $x = 2$, $x = 3$

$\therefore f(x)$ has two critical points $x = 2$, $x = 3$

$\because f''(x) = 12x - 30$

$\therefore f''(2) = 24 - 30 = -6 < 0 , f''(3) = 36 - 30 = 6 > 0$

$\therefore (2, f(2))$ is the point of local maximum value

$, \because f'(x) = 6x^2 - 30x + 36$

$\therefore$ Equation of the curve is $f(x) = \int f'(x)\, dx$

i.e. $f(x) = \int (6x^2 - 30x + 36)\, dx$

$f(x) = 2x^3 - 15x^2 + 36x + c$

$\because f(2) = 28$ $\qquad \therefore 28 = 16 - 60 + 72 + c$

$c = 0$

$\therefore$ Equation of the required curve :

$y = 2x^3 - 15x^2 + 36x$

13 (a)

Solution :

$y = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \dots + \infty$

$\therefore y = e^x , y' = e^x = y , y'' = e^x = y$

$\therefore 2y'' + 3y' - 4y = 2e^x + 3e^x - 4e^x$

$= e^x = y$

14 (a)

Solution :

$y = e^x \sin x$

$\therefore \frac{dy}{dx} = e^x \cos x + \sin x \times e^x$

$= e^x (\sin x + \cos x)$

$, \frac{d^2y}{dx^2} = e^x (\cos x - \sin x) + (\sin x + \cos x) \times e^x$

$= e^x (2 \cos x)$

$\therefore \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y$

$= e^x \times 2\cos x - 2 \times e^x (\sin x + \cos x) + 2 \times e^x \sin x$

$= 2\cos x \times e^x - 2\sin x \times e^x$

$- 2\cos x \times e^x + 2\sin x \times e^x =$ zero.

15 (b)

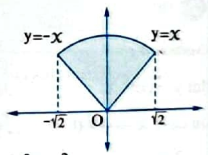
Solution :

$\int \frac{\sin^6 x}{\cos^8 x} dx = \int \frac{\sin^6 x}{\cos^6 x} \times \frac{1}{\cos^2 x} dx$

$= \int \tan^6 x \times \sec^2 x\, dx = \frac{1}{7} \tan^7 x + c$

16 (a)

Solution :



Points of intersection from equation of any of the two straight lines , then $x^2 = y^2$

, then substitute in the equation of the circle

$\therefore x^2 + x^2 = 4$

$\therefore 2x^2 = 4$

$x = \pm\sqrt{2}$

$\therefore v = \pi \int_{-\sqrt{2}}^{\sqrt{2}} (y_2^2 - y_1^2)\, dx$

$= 2\pi \int_0^{\sqrt{2}} (4 - x^2 - x^2)\, dx$

$= 2\pi \int_0^{\sqrt{2}} (4 - 2x^2)\, dx = 2\pi \left[4x - \frac{2}{3}x^3\right]_0^{\sqrt{2}}$

$= \frac{16\sqrt{2}}{3} \pi$ cubic unit.

17 (d)

Solution :

$\because f(x) = x^3 - a x^2 + b$ and its domain $= \mathbb{R}$

$\therefore f'(x) = 3x^2 - 2ax$

$\because$ The function has local minimum value at $x = 2$

$\therefore f'(2) = 0$ $\qquad \therefore 3(2)^2 - 2a(2) = 0$

$\therefore a = 3$

$\because (2, 4) \in$ the function f $\qquad \therefore f(2) = 4$

$\therefore (2)^3 - 3(2)^2 + b = 4$ $\qquad \therefore b = 8$

$\therefore a \times b = 24$

18 (b)

19 (b)

Solution :

Let the number of extra appliances $= x$ set

$\therefore$ The total number of appliances $= (50 + x)$ set

$\therefore$ The profit in each appliances $= 30 - \frac{1}{2}x$ L.E.

$\therefore$ The profit $P = (50 + x)\left(30 - \frac{1}{2}x\right)$

$= 1500 + 5x - \frac{1}{2}x^2$

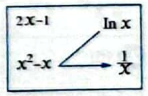
$P' = 5 - x$ $\qquad$ putting $P' = 0$ $\qquad \therefore x = 5$

$, \because P'' = -1 < 0$ (maximum value)

$\therefore$ The number of appliances to get maximum profit $= 55$ appliances.

20 (c)

Solution :



$\int (2x - 1) \ln x\, dx$

$= (x^2 - x) \ln x - \int \frac{1}{x}(x^2 - x)\, dx$

$\therefore y z = (x^2 - x) \ln x$

21 (b)

Solution :

$\because y_1 = x^3 , y_2 = 4x$ $\qquad$ put $y_1 = y_2$

$\therefore x^3 = 4x$ $\qquad \therefore x(x^2 - 4) = 0$

$\therefore x = 0$ or $x = 2$ or $x = -2$

$\therefore$ The area of the shaded region

$$= \int_{-2}^{0} (y_1 - y_2)\, dx + \int_{0}^{2} (y_2 - y_1)\, dx$$

$$= \int_{-2}^{0} (x^3 - 4x)\, dx + \int_{0}^{2} (4x - x^3)\, dx$$

$$= \left[\frac{1}{4}x^4 - 2x^2\right]_{-2}^{0} + \left[2x^2 - \frac{1}{4}x^4\right]_{0}^{2}$$

$= [\text{zero} - (-4)] + [4 - \text{zero}] = 8$ square unit.

---

22 (d)

Solution :

$$\int \frac{1}{\cos x - 1}\, dx = \int \frac{1}{\cos x - 1} \times \frac{\cos x + 1}{\cos x + 1}\, dx$$

$$= \int \frac{\cos x + 1}{\cos^2 x - 1}\, dx = \int \frac{\cos x + 1}{-\sin^2 x}\, dx$$

$$= \int (-\csc x \cot x - \csc^2 x)\, dx = \csc x + \cot x + c$$

---

23 (a)

Solution :

$\because y = \sin x - \cos x$

$\therefore \frac{dy}{dx} = \cos x + \sin x$

$\therefore \frac{d^2y}{dx^2} = -\sin x + \cos x$

$\therefore \frac{d^3y}{dx^3} = -\cos x - \sin x$

$\therefore \frac{d^4y}{dx^4} = \sin x - \cos x = y \qquad \therefore \frac{d^{16}y}{dx^{16}} = y$

$\therefore \frac{d^{17}y}{dx^{17}} = \frac{dy}{dx} = \cos x + \sin x$

---

24 (d)

Solution :

$\because \sin(xy) + \cos(xy) = \text{zero}$

$\therefore \sin(xy) = -\cos(xy)$

$\therefore \tan(xy) = -1 \qquad$ (Differentiate with respect to x)

$\therefore \sec^2(xy) \times (y + x\grave{y}) = \text{zero}$

$\because \sec^2(xy) \neq 0$

$\therefore y + x\grave{y} = 0$, hence $\grave{y} = \frac{-y}{x}$

---

25 (a)

Solution :

$\because y^2(1 + x^2) = 8$

$\therefore 2y\frac{dy}{dx}(1 + x^2) + y^2(2x) = \text{zero}$

at the point $(-1, 2)$

$8\frac{dy}{dx} + (-8) = 0 \qquad \therefore \frac{dy}{dx} = 1$

$\because$ Slope of tangent $= 1$

$\therefore$ Slope of normal $= -1$

$\therefore$ Equation of normal at $(-1, 2)$ is

$y - 2 = -1(x + 1)$

$\therefore x + y - 1 = 0$

Exam 14

1 (a)

Solution :

$$\int_{1}^{4} f(x)\, dx + \int_{4}^{8} f(x)\, dx = \int_{1}^{8} f(x)\, dx$$

$\therefore 2b = 4 \qquad \therefore b = 2$

---

2 (b)

Solution :

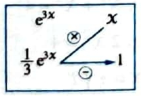
let $y = \cot^2 x \qquad \therefore \tan^2 x = \frac{1}{y}$

in case $x \longrightarrow 0 \qquad \therefore y \longrightarrow \infty$

$$\therefore \lim_{x \to 0} (1 + 3\tan^2 x)^{\cot^2 x} = \lim_{y \to \infty} \left(1 + \frac{3}{y}\right)^y = e^3$$

---

3 (d)



Solution :

$\frac{dy}{dx} = x e^{3x}$

$\therefore y = \int x e^{3x}\, dx\ \frac{1}{3} x e^{3x} - \frac{1}{3}\int e^{3x}\, dx$

$\therefore y = \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + c$

$\therefore \left(\frac{1}{3}, 5\right)$ lies on the curve

$\therefore 5 = \frac{1}{3}\left(\frac{1}{3}\right)e - \frac{1}{9}e + c \qquad \therefore c = 5$

$\therefore y = \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + 5$

4 (c)

Solution :

$$\frac{dy_1}{dx} \div \frac{dy_2}{dx} = \frac{3 \times \frac{1}{2\sqrt{x+1}}}{3\sqrt{x+1}} \div \frac{5 \times \frac{1}{2\sqrt{x+1}}}{5 \times \sqrt{x+1}} = 1$$

---

5 (a)

Solution :

area of the triangle (A) $= \frac{1}{2} \times 6 \times 6 \times \sin x$

$\therefore A = 18 \sin x$

$\therefore \frac{dA}{dt} = 18\cos x \frac{dx}{dt} = 18\cos\left(\frac{\pi}{6}\right) \times \frac{\pi}{90} = \frac{\sqrt{3}}{10}\pi$

---

6 (c)

Solution :

$$\int (\sin^2 x + \cos^2 x + \cot^2 x)\, dx = \int (1 + \cot^2 x)\, dx$$

$$= \int \csc^2 x\, dx$$

$$= -\cot x + C$$

---

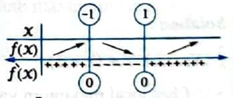
7 (d)

Solution :

$y = x^3 - 3x + 4$

$\therefore \grave{y} = 3x^2 - 3$

Put $\grave{y} = 0$



$\therefore x = \pm 1$

$\therefore f$ is decreasing in $]-1, 1[$

---

8 (b)

Solution :

$$y = \begin{cases} x^2 & , \quad x \geq 0 \\ -x^2 & , \quad x < 0 \end{cases}$$

$$, \grave{y} = \begin{cases} 2x & , \quad x \geq 0 \\ -2x & , \quad x < 0 \end{cases}$$

$\left(\frac{dy}{dx}\right)_{x=-2} = -2 \times -2 = 4$

slope of the normal $= \frac{-1}{4}$

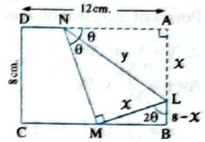
$\therefore$ Equation of the normal $= \frac{y + 4}{x + 2} = \frac{-1}{4}$

i.e. $4y + x + 18 = 0$

---

9 (a)

Solution :



In Δ LBM :

$\cos 2\theta = \frac{8 - x}{x} \qquad (1)$

In Δ ALN :

$\sin\theta = \frac{x}{y}$

$\because \cos 2\theta = 1 - 2\sin^2\theta$

$= 1 - 2\left(\frac{x}{y}\right)^2 \qquad (2)$

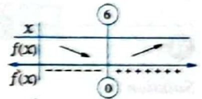
From (1), (2) : $\frac{8 - x}{x} = 1 - 2\left(\frac{x}{y}\right)^2$

$\therefore y^2 = \frac{x^3}{x - 4} \qquad$ (by differentiation with respect to x)

$\therefore 2y\frac{dy}{dx} = \frac{(x - 4)(3x^2) - x^3(1)}{(x - 4)^2}$

$= \frac{2x^3 - 12x^2}{(x - 4)^2}$

Put $\frac{dy}{dx} = 0$



$\therefore x = 0$ (refused) $\qquad$ or $x = 6$

$\therefore x = 6$ Makes y is minimum value.

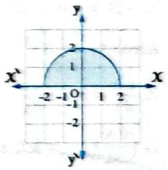
---

10 (c)

Solution :

Let $y = \sqrt{4 - x^2}$

$, y^2 = 4 - x^2$



$\therefore x^2 + y^2 = 4$ is equation

of a circle its radius

$= 2$ length units

$\therefore \pi \int_{-2}^{2} (4 - x^2)\, dx = \pi \int_{-2}^{2} y^2\, dx$

is the volume resulting from rotation about x-axis

a complete revolution , it represents volume of

a sphere its radius $= 2$

---

11 (c)

---

12 (a)

Solution :

$y = \sin(5x^3)\csc(5x^3) = 1 \qquad \therefore \frac{dy}{dx} = \text{zero}$

13 (c)

Solution :

Points of intersection with x-axis : $x^2 - 9 = 0$

$\therefore x = 3 \;,\; x = -3$

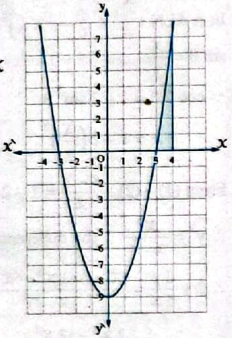
$$\text{Area} = \int_3^4 y \, dx$$

$$= \int_3^4 (x^2 - 9) \, dx$$

$$= \left[\frac{1}{3} x^3 - 9x\right]_3^4$$

$$= \left[\frac{-44}{3}\right] - [-18]$$

$$= \frac{10}{3} \text{ square unit}$$



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14 (a)

Solution :

$\frac{dV}{dt} = -20 \text{ cm}^3/\text{sec.}$

$\because V = \frac{4}{3} \pi r^3 \qquad \therefore \frac{dV}{dt} = 4 \pi r^2 \frac{dr}{dt}$

when r = 10

$\therefore -20 = 4 \pi \times 10^2 \times \frac{dr}{dt} \qquad \therefore \frac{dr}{dt} = \frac{-1}{20\pi} \text{ cm/sec.}$

Area (A) $= 4 \pi r^2$

$\therefore \frac{dA}{dt} = 8 \pi r \frac{dr}{dt}$

$\therefore \left(\frac{dA}{dt}\right)_{r=10} = 8 \pi \times 10 \times \frac{-1}{20\pi} = -4 \text{ cm}^2/\text{sec.}$

---

15 (c)

Solution :

$\because \sqrt{x} + \sqrt{y} = 1$, then $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = \text{zero}$

$\therefore \frac{dy}{dx} = \frac{-\sqrt{y}}{\sqrt{x}} \qquad \therefore \left[\frac{dy}{dx}\right]_{\left(\frac{1}{4}, \frac{1}{4}\right)} = -1$

---

16 (d)

Solution :

The 5^{th} derivative of any term its degree is less than 5 = zero

$\therefore$ The 5^{th} derivative of the term

$a x^5 = a \times 5 \times 4 \times 3 \times 2 \times 1 = a \lfloor 5$ when $a \neq 0$

$\therefore$ The 5^{th} derivative = (constant $\neq 0$)

---

17 (a)

Solution :

$$\int \frac{\ln x^5}{x \ln x^3} \, dx = \int \frac{5 \ln x}{3 x \ln x} \, dx = \int \frac{5}{3} \times \frac{1}{x} \, dx$$

$$= \frac{5}{3} \ln |x| + c$$

---

18 (b)

---

19 (b)

Solution :

$\because a^y = b^x$ by taking natural logarithm of both sides

$\therefore y \ln a = x \ln b$

by differentiating both sides with respect to x

$\therefore \frac{dy}{dx} \ln a = \ln b$

$\therefore \frac{dy}{dx} = \frac{\ln b}{\ln a} = \log_a b$

---

20 (b)

Solution :

$\acute{f}(x) = 2kx + (k+5)$

, $\because f$ has local maximum value at $x = 2$

$\therefore \acute{f}(2) = 0$

$\therefore 4k + k + 5 = 0 \qquad \therefore k = -1$

---

21 (b)

Solution :

$\because \int_a^b (3x^2 - 1) \, dx = \text{zero} \qquad \therefore [x^3 - x]_a^b = 0$

$\therefore (b^3 - b) - (a^3 - a) = 0$

$\therefore (b^3 - a^3) - (b - a) = 0$

$\therefore (b - a)(b^2 + ab + a^2) - (b - a) = 0$

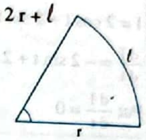
$\therefore (b - a)[b^2 + a^2 + ab - 1] = 0$

$\therefore b \neq a \qquad \therefore b^2 + a^2 + ab - 1 = 0$

$\therefore b^2 + a^2 = 1 - ab$

22 (d)

Solution :



$\because$ The perimeter of the sector (p) $= 2r + \ell$

$\therefore \ell = p - 2r$

$\therefore \text{Area} = \frac{1}{2} \ell r$

$\therefore A = \frac{1}{2}(p - 2r) \times r = \frac{1}{2} pr - r^2$

$\therefore \frac{dA}{dr} = \frac{1}{2} p - 2r$

Put $\frac{dA}{dr} = 0 : \therefore r = \frac{p}{4}$

, $\because \frac{d^2A}{dr^2} = -2 < 0$ for all values of r

$\therefore$ The area is maximum at $r = \frac{p}{4}$

---

23 (a)

Solution :

$\because f(x) = 10 x e^{-x}$

$\therefore \acute{f}(x) = 10 e^{-x} - 10 x e^{-x}$

Put $\acute{f}(x) = 0 : \therefore 10 e^{-x}(1 - x) = 0$

$\therefore x = 1 \in [0, 4]$, $f(1) = \frac{10}{e}$

, $\because f(0) = 0$, $f(4) = \frac{40}{e^4}$

The function has an absolute maximum value $= \frac{10}{e}$ at $x = 1$

---

24 (b)

Solution :

$\because f(x) = \frac{1 - \cot x}{1 + \cot x}$

$$\therefore \acute{f}(x) = \frac{(1 + \cot x)(\csc^2 x) - (1 - \cot x)(-\csc^2 x)}{(1 + \cot x)^2}$$

$$\therefore \acute{f}\left(\frac{\pi}{4}\right) = \frac{(1+1)(2) - (1-1)(-2)}{(1+1)^2} = 1$$

Another solution :

$$f(x) = \frac{1 - \cot x}{1 + \cot x} \times \frac{\tan x}{\tan x} = \frac{\tan x - 1}{\tan x + 1} = \frac{\tan x - \tan \frac{\pi}{4}}{1 + \tan x \tan \frac{\pi}{4}}$$

$$= \tan\left(x - \frac{\pi}{4}\right)$$

$\therefore \acute{f}(x) = \sec^2\left(x - \frac{\pi}{4}\right)$

$\therefore \acute{f}\left(\frac{\pi}{4}\right) = \sec^2 0 = 1$

---

25 (d)

Solution :

The equation of the straight line L_1 is $\frac{x}{1} + \frac{y}{2} = 1$

$\therefore x_1 = 1 - \frac{1}{2} y$

The equation of the straight line L_2 is $\frac{x}{3} + \frac{y}{2} = 1$

$\therefore x_2 = 3 - \frac{3}{2} y$

$$\therefore \text{The volume} = \pi \int_0^2 (x_2^2 - x_1^2) \, dy$$

$$= \pi \int_0^2 \left[\left(3 - \frac{3}{2} y\right)^2 - \left(1 - \frac{1}{2} y\right)^2\right] dy$$

$$= \pi \left[\frac{\left(3 - \frac{3}{2} y\right)^3}{3 \times -\frac{3}{2}} - \frac{\left(1 - \frac{1}{2} y\right)^3}{3 \times -\frac{1}{2}}\right]_0^2$$

$$= \pi \left[\frac{-2}{9}\left(3 - \frac{3}{2} y\right)^3 + \frac{2}{3}\left(1 - \frac{1}{2} y\right)^3\right]_0^2$$

$$= \pi \left[\text{zero} - \left(\frac{-16}{3}\right)\right] = \frac{16}{3} \pi$$

Exam 15

1 (b)

Solution :

$$\lim_{x \to 0} \frac{10^{\sin x} - 1}{\tan x} = \lim_{\sin x \to 0} \frac{10^{\sin x} - 1}{\sin x} \times \lim_{x \to 0} \cos x$$

$$= \ln 10 \times 1 = \ln 10$$

---

2 (a)

Solution :

$$f(x) = |x - 3| = \begin{cases} -x + 3 & , \quad x < 3 \\ x - 3 & , \quad x \geq 3 \end{cases}$$

$$\therefore \int_0^6 |3 - x| \, dx = \int_0^6 |x - 3| \, dx$$

$$= \int_0^3 (-x + 3) \, dx + \int_3^6 (x - 3) \, dx$$

$$= \left[-\frac{1}{2} x^2 + 3x\right]_0^3 + \left[\frac{1}{2} x^2 - 3x\right]_3^6$$

$$= \frac{9}{2} + \frac{9}{2} = 9$$

---

3 (b)

Solution :

$$\acute{f}(x) = \frac{(x^2 + 1) - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

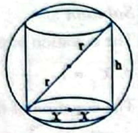
$\because \acute{f}(x) = 0 \qquad \therefore 1 - x^2 = 0$

$x = 1 \in [0, 2]$ or $x = -1 \notin [0, 2]$

$\therefore f(1) = \frac{1}{2}$, $f(0) = \text{zero}$, $f(2) = \frac{2}{5}$

$\therefore$ The absolute maximum value $= \frac{1}{2}$

4 (b)



Solution :

$(2x)^2 + h^2 = (2r)^2$

$\therefore x^2 = \frac{1}{4}(4r^2 - h^2)$

$\therefore$ volume of the cylinder(v)

$= \pi x^2 h = \pi h \times \frac{1}{4}(4r^2 - h^2) = \frac{\pi}{4}(4r^2 h - h^3)$

$\therefore \frac{dv}{dh} = \frac{\pi}{4}(4r^2 - 3h^2)$ when $\frac{dv}{dh} =$ zero

$\therefore 4r^2 - 3h^2 = 0 \qquad \therefore h^2 = \frac{4}{3}r^2 \qquad \therefore h = \frac{2r}{\sqrt{3}}$

$\therefore \frac{d^2v}{dh^2} < 0$ (v is greatest as possible)

---

5 (a)

Solution :

$y = \ln\sqrt{\tan x}$

$\therefore \frac{dy}{dx} = \frac{\frac{\sec^2 x}{2\sqrt{\tan x}}}{\sqrt{\tan x}} \qquad \therefore \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = 1$

---

6 (c)

---

7 (b)

Solution :

$\int \frac{\ln x}{x}\,dx = \int \ln x \times \frac{1}{x}\,dx = \frac{(\ln x)^2}{2} + c$

---

8 (b)

Solution :

$x = a\sec^2\theta$

$\therefore \frac{dx}{d\theta} = 2a\sec\theta.\sec\theta.\tan\theta = 2a\sec^2\theta\tan\theta$

, $y = a\tan^3\theta$

$\therefore \frac{dy}{d\theta} = 3a\tan^2\theta.\sec^2\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{3a\tan^2\theta\sec^2\theta}{2a\sec^2\theta\tan\theta} = \frac{3\tan\theta}{2}$

$\frac{d^2y}{dx^2} = \frac{3}{2}\sec^2\theta \times \frac{1}{2a\sec^2\theta\tan\theta} = \frac{3\cot\theta}{4a}$

9 (a)

Solution :

$I = 2\cos t + 2\sin t$

$\frac{dI}{dt} = -2\sin t + 2\cos t$

Put $\frac{dI}{dt} = 0$

$\therefore -2\sin t + 2\cos t = 0$

$\therefore \sin t = \cos t \quad \therefore \tan t = 1$

$\therefore t = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

$\frac{d^2I}{dt^2} = -2\cos t - 2\sin t$

when $t = \frac{\pi}{4}$

$\frac{d^2I}{dt^2} = -2\sqrt{2} < 0$ (Max)

when $t = \frac{5\pi}{4} : \frac{d^2I}{dt^2} = 2\sqrt{2} > 0$ (Min)

$\therefore$ Maximum value of current at $t = \frac{\pi}{4}$

is $I = 2\sqrt{2}$

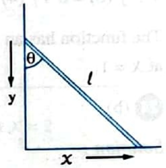
---

10 (d)

Solution :

Let length of the ladder

$= \ell$ length unit

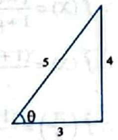


, $\because x^2 + y^2 = \ell^2$

$\therefore 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\frac{dx}{dt} = \frac{-y}{x} \times \frac{dy}{dt}$

When $\csc\theta = \frac{5}{4}$



$x = \ell\sin\theta$

, $y = \ell\cos\theta$

$\therefore \frac{dx}{dt} = \frac{-3}{4} \times -k = \frac{3k}{4}$ length unit/sec. $(k > 0)$

---

11 (d)

Solution :

$\int_{-2}^{5} f(x)\,dx + \int_{-2}^{5} |f(x)|\,dx$

$= A_1 - A_2 + A_3 + A_1 + A_2 + A_3$

$= 2(A_1 + A_3) = 2(5 + 8) = 26$

12 (b)

Solution :

$f(x) = x - x\ln x$

$\therefore f'(e) = 1 - x\left(-\frac{1}{x}\right) - \ln x$

$f'(e) = 1 - 1 - \ln e = -1$

---

13 (c)

Solution :

Find equation of the tangent at (2 , 2)

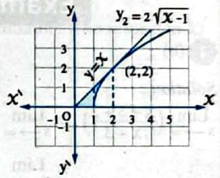
$\frac{dy}{dx} = \frac{2}{2\sqrt{x-1}} = \frac{1}{\sqrt{x-1}}$

at $x = 2 : \frac{dy}{dx} = 1$

$\therefore$ Equation of the tangent :

$y - 2 = x - 2$

$\therefore y = x$



$\therefore$ Points of intersection

of the tangent with

x-axis (0 , 0)

$\therefore v = v_1 + v_2 = \pi \int_0^1 y_1^2\,dx + \pi\int_1^2 (y_1^2 - y_2^2)\,dx$

$= \pi\int_0^1 x^2\,dx + \pi\int_1^2 \left[x^2 - \left(2\sqrt{x-1}\right)^2\right]dx$

$= \pi\left[\frac{x^3}{3}\right]_0^1 + \pi\int_1^2 [x^2 - 4(x-1)]\,dx$

$= \pi\left[\frac{1}{3}\right] + \pi\int_1^2 (x^2 - 4x + 4)\,dx$

$= \frac{\pi}{3} + \pi\left[\frac{1}{3}x^3 - 2x^2 + 4x\right]_1^2$

$= \frac{\pi}{3} + \pi\left[\left(\frac{8}{3}\right) - \left(\frac{7}{3}\right)\right] = \frac{2\pi}{3}$ cubic unit.

---

14 (a)

Solution :

$y = x\ln x$

$\therefore \frac{dy}{dx} = x \times \frac{1}{x} + \ln x = 1 + \ln x$

$\because$ slope of the given straight line = 1

$\therefore 1 + \ln x = -1 \qquad \therefore \ln x = -2 \qquad \therefore x = e^{-2}$

$\therefore y = e^{-2}.\ln e^{-2} = -2e^{-2}$

$\therefore$ The point is $(e^{-2}, -2e^{-2})$

$\therefore$ The equation of the normal : $\frac{y + 2e^{-2}}{x - e^{-2}} = 1$

$\therefore x - y = 3e^{-2}$

---

15 (b)

Solution :

$\int \frac{\sec^2 x}{\tan x}\,dx = \ln|\tan x| + c$

---

16 (a)

Solution :

let $y = e^{\sin x} \qquad \therefore \frac{dy}{dx} = e^{\sin x}.\cos x$

, $z = \sin x \qquad \therefore \frac{dz}{dx} = \cos x$

$\therefore \frac{dy}{dz} = \frac{dy}{dx} \div \frac{dz}{dx} = \frac{e^{\sin x}\cos x}{\cos x} = e^{\sin x}$

---

17 (d)

---

18 (d)

Solution :

$\because y_1 = x^2 + ax + b \qquad \therefore m_1 = \frac{dy_1}{dx} = 2x + a$

, $\because y_2 = cx - x^2$

$\therefore m_2 = \frac{dy_2}{dx} = c - 2x$

, $\because$ the two curves are touching at (1 , 0)

$\therefore m_1 = m_2$ at $x = 1$

$2 + a = c - 2 \qquad \therefore a - c = -4$ (1)

$\because$ (1 , 0) lies in both the two curves

$\therefore 0 = 1 + a + b$ (2)

, $0 = c - 1 \qquad \therefore c = 1$ (3)

From (1) , (2) , (3) , then $a = -3$, $b = 2$, $c = 1$

$\therefore b + c - a = 2 + 1 - (-3) = 6$

---

19 (a)

Solution :

$\because f'(x) = x.f(x)$

$\therefore f''(x) = f(x) + x.f'(x) = f(x) + x.x\,f(x)$

$\therefore f''(3) = f(3) + 3 \times 3 \times f(3)$

$= -5 + 3 \times 3 \times -5 = -50$

20 (a)

Solution :

$\because f(x) = \cot\left(\frac{\pi}{3}\sin x\right)$

$\therefore f'(x) = \left(-\frac{\pi}{3}\cos x\right)\left(\csc^2\left(\frac{\pi}{3}\sin x\right)\right)$

$\therefore f'\left(\frac{\pi}{6}\right) = -\frac{\pi}{3} \times \frac{\sqrt{3}}{2} \times 4 = \frac{-2\sqrt{3}\pi}{3}$

---

21 (b)

Solution :

$\because (3, -9) \in$ the curve

$\therefore -9 = (3)^3 + a(3)^2 + b(3)$

$\therefore 9a + 3b = -36 \qquad \therefore 3a + b = -12 \qquad (1)$

$\therefore f(x) = x^3 + ax^2 + bx$

$\therefore f'(x) = 3x^2 + 2ax + b \qquad \therefore f''(x) = 6x + 2a$

The function has an inflection point at $x = 3$

$\therefore 6 \times 3 + 2a = \text{zero} \qquad \therefore a = -9$

From (1) : $3(-9) + b = -12$

$\therefore b = 15 \qquad \therefore a + b = -9 + 15 = 6$

---

22 (d)

Solution :

$$\int_0^1 f'(x) \cdot f(x)\, dx = \left[\frac{(f(x))^2}{2}\right]_0^1$$

$$= \frac{1}{2}\left[(f(1))^2 - (f(0))^2\right]$$

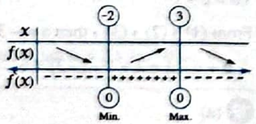
$$= \frac{1}{2}\left[3^2 - 4^2\right] = \frac{-7}{2}$$

---

23 (b)

Solution :

$\therefore$ (b) is false



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24 (d)

Solution :

f increases when f' is positive

$\therefore f$ increases at $x \in]1, 3[$

25 (b)

Solution :

$\because$ The quadratic function is symmetric about its axis of symmetry

, then $k = \frac{0+6}{2} = 3$ and its equation $f(x)$

$= a(x-3)^2 + 9$

, $\because (6, 0) \in$ the function

$\therefore a = -1 \qquad \therefore f(x) = -x^2 + 6x$

$\therefore$ The area of the shaded region

$$= \int_0^3 \left[9 - (-x^2 + 6x)\right] dx$$

$$= \int_0^3 (x-3)^2\, dx = \left[\frac{1}{3}(x-3)^3\right]_0^3$$

$= \text{zero} + 9 = 9$ square units.

Exam 16

1 (b)

Solution :

$$\lim_{x \to \infty}\left(\frac{x+3+2}{x+3}\right)^x = \lim_{x \to \infty}\left(1 + \frac{2}{x+3}\right)^x$$

$$= \lim_{x \to \infty}\left(1 + \frac{2}{x+3}\right)^{x+3} \times \lim_{x \to \infty}\left(1 + \frac{2}{x+3}\right)^{-3}$$

$$= e^2 \times 1^{-3} = e^2$$

---

2 (b)

Solution :

$\frac{d^2y}{dx^2} = 6 - 12x$

$\therefore \frac{dy}{dx} = \int (6 - 12x)\, dx = 6x - 6x^2 + C_1$

$\because$ has critical point at $x = 1$

$\therefore \text{zero} = 6 - 6 + C_1 \qquad \therefore C_1 = 0$

$\frac{dy}{dx} = 6x - 6x^2 \qquad (1)$

Put $\frac{dy}{dx} = \text{zero} \qquad \therefore x = 0$ or 1

At $x = 1$

then $\frac{d^2y}{dx^2} < 0$ has maximum value

At $x = 0$ then $\frac{d^2y}{dx^2} > 0$ has minimum value $= 4$

$\therefore (0, 4)$ lies on the curve.

$\because y = \int (6x - 6x^2)\, dx = 3x^2 - 2x^3 + C_2$

$\therefore 4 = 3(0) - 2(0) + C_2 \qquad \therefore C_2 = 4$

$\therefore y = 3x^2 - 2x^3 + 4$

$\therefore$ Slope of the tangent at $x = -1$

$\left(\frac{dy}{dx}\right)_{x=-1} = 6(-1) - 6(-1)^2 = -12$

$\therefore$ Slope of the normal $= \frac{1}{12}$ at $x = -1$

$\therefore y = 3(-1)^2 - 2(-1)^3 + 4 = 9$

$\therefore$ The point is $(-1, 9)$

$\therefore$ Equation of the normal $\frac{y-9}{x+1} = \frac{1}{12}$

$\therefore x - 12y + 109 = 0$

---

3 (a)

Solution :

$\sin 2x = \tan y$ (by differentiating with respect to x)

$\therefore 2\cos 2x = \sec^2 y \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{2\cos 2x}{\sec^2 y}$

$\left(\frac{dy}{dx}\right)_{\left(\frac{3\pi}{4}, \frac{3\pi}{4}\right)} = \text{zero}$

$\therefore$ measure of the angle $= 0°$

---

4 (b)

Solution :

$y = x^2 + 3x + 2$

$\frac{dy}{dx} = 2x + 3$

$z = 3x^2 - 5x + 4 \quad , \quad \frac{dz}{dx} = 6x - 5$

$\frac{dy}{dz} = \frac{dy}{dx} \div \frac{dz}{dx} = \frac{2x+3}{6x-5}$

$\frac{d^2y}{dz^2} = \frac{2(6x-5) - 6(2x+3)}{(6x-5)^2} \times \frac{1}{6x-5} = \frac{-28}{(6x-5)^3}$

$\left(\frac{d^2y}{dz^2}\right)_{x=2} = \frac{-28}{343} = \frac{-4}{49}$

---

5 (a)

Solution :

$$\int_0^2 [f(x)]^2 f'(2)\, dx = \left[\frac{1}{3}[f(x)]^3\right]_0^2$$

$$= \frac{1}{3}(f(2))^3 - \frac{1}{3}(f(0))^3$$

$$= \frac{1}{3} \times 0 - \frac{1}{3} \times 3^3 = -9$$

6 (b)

Solution :

let the dimensions of the rectangle are x , y

$\therefore x + y = 20$

$\therefore$ Area of the rectangle $(A) = xy$

$\therefore A = x(20 - x) = 20x - x^2$

$\frac{dA}{dx} = 20 - 2x$, Put $\frac{dA}{dx} = 0$

$\therefore 20 - 2x = 0 \qquad \therefore x = 10$

$\therefore y = 10$, $\frac{d^2A}{dx^2} = -2 < 0$

i.e. The area is Max.area.

---

7 (b)

Solution :

$$\int_2^3 2x\, g(x^2) \cdot g'(x^2)\, dx = \left[\frac{1}{2}[g(x^2)]^2\right]_2^3$$

$$= \frac{1}{2}\left[(g(9))^2 - ((g(4))^2\right]$$

$$= \frac{1}{2}(49 - 9) = 20$$

---

8 (a)

Solution :

$\because A = \frac{1}{2}ab$

$\therefore \text{zero} = \frac{1}{2}a\frac{db}{dt} + \frac{1}{2}b\frac{da}{dt}$

$\therefore \frac{da}{dt} = \frac{-a}{b} \times \frac{db}{dt}$

, $\because A = 24\ \text{cm}^2 \qquad \therefore ab = 48$

at $b = 8$ cm. , then $a = 6$ cm.

$\therefore \frac{da}{dt} = \frac{-6}{8} \times 1 = \frac{-3}{4}$ cm./sec.

---

9 (a)

Solution :

$f(x) = x^{2x}$

(by taking logarithmic both two sides to the base e)

$\therefore \ln f(x) = 2x \ln x$

(by differentating both sides with respect to x)

$\therefore \frac{f'(x)}{f(x)} = 2x \times \frac{1}{x} + 2\ln x$

$\therefore f'(x) = x^{2x}[2 + 2\ln x]$

$\therefore f'(e) = e^{2e}(2 + 2) = 4e^{2e}$

10 (c)

Solution :

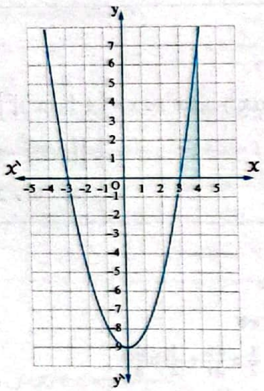
$$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{-1} (2x^3+3)\,dx + \int_{-1}^{2} (3x+4)\,dx$$

$$= \left[\frac{1}{2}x^4 + 3x\right]_{-2}^{-1} + \left[\frac{3}{2}x^2 + 4x\right]_{-1}^{2}$$

$$= \left[\frac{-5}{2} - 2\right] + \left[14 - \left(\frac{-5}{2}\right)\right] = 12$$

11 (a)

Solution :



To find points of intersection with x-axis

$x^2 - 9 = 0 \qquad \therefore x = \pm 3$

$\therefore \int_{3}^{4} (x^2-9)\,dx = \left[\frac{x^3}{3} - 9x\right]_3^4 = \frac{10}{3}$

$\therefore$ Area $= \frac{10}{3}$ square unit.

12 (c)

13 (d)

Solution :

$$\int \frac{\cos^2 x}{1-\sin x}\,dx = \int \frac{1-\sin^2 x}{1-\sin x}\,dx$$

$$= \int \frac{(1-\sin x)(1+\sin x)}{(1-\sin x)}\,dx$$

$$= \int (1+\sin x)\,dx = x - \cos x + c$$

14 (c)

Solution :

$f(x) = x\sin x + \cos x$

$\therefore f'(x) = x\cos x + \sin x - \sin x = x\cos x$

, put $f'(x) = 0 \qquad \therefore x\cos x = 0$

$\therefore x = 0$ or $\cos x = 0 \qquad \therefore x = \frac{\pi}{2}$

$\therefore f(0) = 1$, $f(\pi) = -1$, $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

$\therefore$ The function has absolute minimum value at $x = \pi$

15 (a)

Solution :

Volume $= \pi \int_{-1}^{k} y^2\,dx = \pi \int_{-1}^{k} (e^x)^2\,dx$

$= \pi \int_{-1}^{k} e^{2x}\,dx = \frac{\pi}{2}\left[e^{2x}\right]_{-1}^{k}$

$= \frac{\pi}{2}\left[e^{2k} - e^{-2}\right] = \frac{\pi}{2}\left(e^{10} - e^{-2}\right)$

$\therefore 2k = 10 \qquad \therefore k = 5$

16 (c)

Solution :

$\because y = \sqrt{x + \sec x}$

$\therefore \frac{dy}{dx} = \frac{1 + \sec x \tan x}{2\sqrt{x + \sec x}}$

$\therefore$ The slope of the tangent to the curve at $x = 0$

equals $\left[\frac{dy}{dx}\right]_{x=0} = \frac{1}{2}$

17 (b)

Solution :

$f(x) = 20\,x^{n-1}$

$\therefore f'(x) = 20\,(n-1)\,x^{n-2}$

$\therefore f''(x) = 20\,(n-1)(n-2)\,x^{n-3}$

$\therefore f'''(x) = 20\,(n-1)(n-2)(n-3)\,x^{n-4}$

$\because f^{(4)}(x) = c$ (costant) $\qquad \therefore n - 4 =$ zero $\qquad \therefore n = 4$

$\therefore f^{(4)}(x) = 20 \times 3 \times 2 \times 1 = 120$

$\therefore c + n = 120 + 4 = 124$

18 (c)

Solution :

$\because f(x) = \pi^x e^x$

$\therefore f(x) = (\pi e)^x$

$\therefore f'(x) = (\pi e)^x \ln(\pi e) = f(x) \,.\, \ln(\pi e)$

19 (b)

Solution :

Let the rate of increase of water volume $= \frac{dv}{dt} =$ a "constant"

$\therefore v = \pi r^2 h$ where h is the water height in the cylinder

$\therefore \frac{dv}{dt} = \pi r^2 \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{a}{\pi r^2}$ and it is positive constant quantity

$\therefore$ The curve of the relation between h and t is a straight line with a positive slope.

20 (b)

Solution :

$\because$ Equation of the normal is $\qquad y = \frac{-1}{3}x + \frac{3}{2}$

$\therefore f(1) = g(1) = \frac{-1}{3}(1) + \frac{3}{2} = \frac{7}{6}$

, $\because$ slope of normal $= \frac{-1}{3}$

$\therefore$ Slope of tangent $= 3 \qquad \therefore f'(1) = g'(1) = 3$

$(f \times g)'(x) = f(x) \,.\, g'(x) + f'(x) \,.\, g(x)$

$\therefore (f \times g)'(1) = \frac{7}{6} \times 3 + 3 \times \frac{7}{6} = 7$

21 (d)

22 (a)

Solution :

$\int \frac{4\ln x}{\ln x}\,dx = \int 4\,dx = 4x + c$

23 (a)

24 (c)

Solution :

Let $a = x \qquad \therefore b = 3x - x^2$

$\therefore s = a + b = x + 3x - x^2 = 4x - x^2$

$\therefore \frac{ds}{dx} = 4 - 2x$

Put $\frac{ds}{dx} = 0 : \therefore x = 2$

, $\because \frac{d^2 s}{dx^2} = -2 < 0$ for all values of x

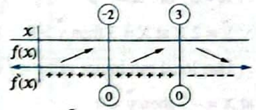
$\therefore a + b$ is maximum at $x = 2$

$[a+b]_{x=2} = 4(2) - (2)^2 = 4$

25 (d)

Solution :

(d) is false because



$f(x)$ increases on $]-\infty , -2[$

Exam 17

1 (b)

Solution :

$$\int_3^5 [4f(x) - 1]\,dx = 4\int_3^5 f(x)\,dx - \int_3^5 dx$$

$$= 4 \times 6 - [x]_3^5$$

$$= 24 - (5-3) = 22$$

2 (b)

Solution :

$f(x) = e^{\tan x} \qquad \therefore f'(x) = e^{\tan x}\sec^2 x$

$\therefore \lim\limits_{x \to \frac{\pi}{4}} \frac{f(x) - f\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = f'\left(\frac{\pi}{4}\right) = e \times 2 = 2e$

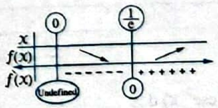
3 (c)

Solution :

$f(x) = x\ln x$

$\therefore f'(x) = x \times \frac{1}{x} + \ln x = 1 + \ln x$

, Put $f'(x) = 0 \qquad \therefore \ln x = -1 \qquad \therefore x = \frac{1}{e}$



The function has local minimum value at $\left(x = \frac{1}{e}\right)$

$f\left(\frac{1}{e}\right) = \frac{1}{e}\ln\frac{1}{e} = \frac{-1}{e}$

4 (a)

Solution :

$y = \frac{1}{3}x^3 - 9x + 2$

$\therefore y' = x^2 - 9$

, at $y' = 0$ $\quad \therefore x^2 - 9 = 0$

$\therefore x = 3$ or $x = -3$

$\because y'' = 2x$ at $x = 3$ then $y'' > 0$

$\therefore$ Has local minimum value $= f(3) = -16$

at $x = -3$ then $y'' < 0$

$\therefore$ Has local maximum value $= f(-3) = 20$

5 (d)

Solution :

$\sin x = \cos x \qquad \therefore \tan x = 1$

$\therefore x = \frac{\pi}{4} + \pi n \qquad \therefore x = 45°$ or 225 , ...

$\therefore$ The area $= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx$

$+ \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\, dx$

$= [\sin x + \cos x]_0^{\frac{\pi}{4}} + [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= (\sqrt{2} - 1) + (-1 + \sqrt{2})$

$= 2\sqrt{2} - 2$ square units.

6 (c)

Solution :

$y = x^{2019}$

$y' = 2019\, x^{2018}$

$y'' = 2019 \times 2018\, x^{2017}$

$y^{(3)} = 2019 \times 2018 \times 2017 \times x^{2016}$

$y^{(2019)} = 2019 \times 2018 \times \ldots \times 3 \times 2 \times 1 = \underline{|2019}$

7 (c)

Solution :

$\int e^x (\cot x - \csc^2 x)\, dx$

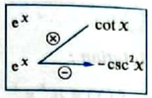
$= \int e^x \cot x\, dx$

$- \int e^x \csc^2 x\, dx$

$= e^x \cot x + \int e^x \csc^2 x\, dx$

$- \int e^x \csc^2 x\, dx$

$= e^x \cot x + c$



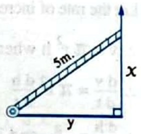
8 (a)

Solution :

$\frac{dx}{dt} = 1$ m./min.

y is length of projection

of the rod on the ground



$x^2 + y^2 = 25$

$2x \frac{dx}{dt} + 2y \frac{dy}{dt} =$ zero

$\frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt}$

When $x = 3$ m. , $y = 4$ m.

$\therefore \frac{dy}{dt} = \frac{-3}{4} \times 1 = \frac{-3}{4}$ m./min.

$\therefore$ The rate of decreasing the projection length $= \frac{3}{4}$ m./nin.

9 (d)

Solution :

$\because y = (\cos x)^{\cos x}$

$\therefore \ln y = \cos x \ln (\cos x)$

(by differentiating both sides with respect to x)

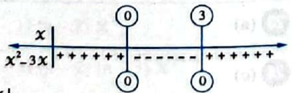
$\therefore \frac{1}{y} y' = \cos x \times \frac{-\sin x}{\cos x} - \sin x \,.\, \ln (\cos x)$

$\therefore y' = (\cos x)^{\cos x} . (-\sin x - \sin x \,.\, \ln \cos x)$

$\therefore f'(0) = 1$ (zero − zero) = zero

10 (b)

Solution :



$f(x) = |x^2 - 3x|$

$= \begin{cases} x^2 - 3x & , \ x \le 0 \\ -x^2 + 3x & , \ 0 < x < 3 \\ x^2 - 3x & , \ x \ge 3 \end{cases}$

$\therefore \int_{-2}^{4} f(x)\, dx = \int_{-2}^{0} (x^2 - 3x)\, dx$

$+ \int_0^3 (-x^2 + 3x)\, dx$

$+ \int_3^4 (x^2 - 3x)\, dx$

$= [\frac{1}{3}x^3 - \frac{3}{2}x^2]_{-2}^{0}$

$+ [-\frac{1}{3}x^3 + \frac{3}{2}x^2]_0^3$

$+ [\frac{1}{3}x^3 - \frac{3}{2}x^2]_3^4$

$= \frac{26}{3} + \frac{9}{2} + \frac{11}{6} = 15$

11 (d)

Solution :

$f(x) = x^3 + 4x + 8$

$f'(x) = 3x^2 + 4$

$3x^2 + 4 > 0 \qquad$ for every $x \in \mathbb{R}$

$\therefore$ The function is increasing on $\mathbb{R}$

12 (b)

Solution :

$y = \sec^n x$

$\therefore \frac{dy}{dx} = n \sec^{n-1}(x) \,.\, \sec x \tan x = n \sec^n x \tan x$

$= n\, y \tan x$

13 (b)

Solution :

$y_1 = 12 - x^2 \quad , \quad 0 < x < 2\sqrt{3}$

$y_2 = x^2 - 12 \quad , \quad 0 < x < 2\sqrt{3}$

$\therefore$ Dimensions of the rectangle $= 2x$ horizontally.

and vertically $y_1 - y_2 = (12 - x^2) - (x^2 - 12)$

$= 12 - x^2 + 12 - x^2$

$= 24 - 2x^2$

$\therefore$ Area of the rectangle $= A = 2x(24 - 2x^2)$

$= 48x - 4x^3$

$\therefore A' = 48 - 12x^2$

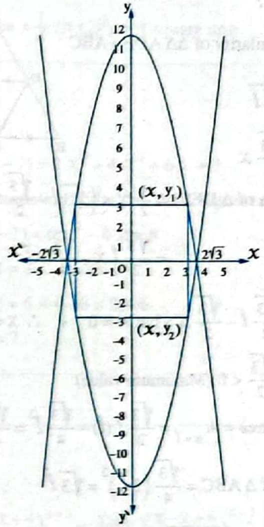
, put $A' = 0$, $x^2 = 4 \quad , \quad \therefore x = 2$

, $A'' = -24x$

$\therefore A''_{(x=2)} < 0$ at $x = 2$ then A has maximum value.

$\therefore$ Maximum area of the triangle A

$= 48 \times 2 - 4 \times 2^3 = 64$ square unit



14 (a)

Solution :

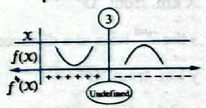
$f(x) = \sqrt[3]{x - 3} = (x - 3)^{\frac{1}{3}} \qquad \therefore f'(x) = \frac{1}{3}(x - 3)^{\frac{-2}{3}}$

$\therefore f''(x) = \frac{-2}{9}(x - 3)^{\frac{-5}{3}} = \frac{-2}{9} \times \frac{1}{\sqrt[3]{(x - 3)^5}}$

$f''(x)$ undefined when

$x - 3 = 0$

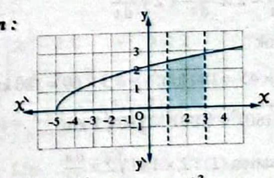
$\therefore x = 3$



The curve is convex upwards in $]3 , \infty[$

15 (a)

Solution :

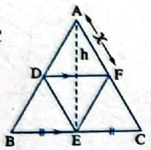


The volume $= \pi \int_1^3 y^2\, dx = \pi \int_1^3 (x + 5)\, dx$

$= \pi [\frac{1}{2}x^2 + 5x]_1^3 = 14\pi$ cubic unit.

16 (a)

Solution :



From similarity of $\Delta\Delta$ ADF , ABC

$$\frac{x}{2\ell}=\frac{h}{\sqrt{3}\,\ell}$$

$$\therefore h=\frac{\sqrt{3}}{2}x$$

$$\therefore A\ (\text{area of } \Delta\, DEF)=\frac{1}{2}x\times\left(\sqrt{3}\,\ell-\frac{\sqrt{3}}{2}x\right)$$

$$=\frac{\sqrt{3}}{2}\ell x-\frac{\sqrt{3}}{4}x^2$$

$$\therefore \grave{A}=\frac{\sqrt{3}}{2}\ell-\frac{\sqrt{3}}{2}x \text{ , put } \grave{A}=0 \quad , \quad \therefore x=\ell$$

$$,\ \grave{\grave{A}}=-\frac{\sqrt{3}}{2}<0 \text{ (Maximum value)}$$

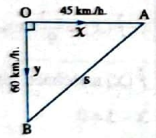
$$\text{greatest area}=A_{x=\ell}=\frac{\sqrt{3}}{2}\ell(\ell)-\frac{\sqrt{3}}{4}\ell^2=\frac{\sqrt{3}}{4}\ell^2$$

$$\therefore \text{area of } \Delta\, ABC=\frac{\sqrt{3}}{4}(2\ell)^2=\sqrt{3}\,\ell^2$$

$$A=\frac{1}{4}\text{ area of } \Delta\, ABC$$

17 (b)

Solution :



Let the 1st train at a distance x km. from "O"

, the 2nd trian at a distance y km. from "O"

and the distance between the two trains (S) km.

$$\therefore S^2=x^2+y^2$$

$$\therefore 2S\frac{ds}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt} \qquad (1)$$

At 3 o'clok

$$\therefore x=4\times 45=180 \text{ km.} \ , \ y=3\times 60=180 \text{ km.}$$

$$\therefore S=\sqrt{(180)^2+(180)^2}=180\sqrt{2} \text{ km.}$$

From equation (1) : $2\times 180\sqrt{2}\times\frac{ds}{dt}$

$$=2(180)(45)+2(180)(60)$$

$$\therefore \frac{ds}{dt}=52.5\sqrt{2} \text{ km/h.}$$

18 (d)

Solution :

$$\lim_{x\to 0}\left(1+\frac{x}{a}\right)^{\frac{a}{x}}=\lim_{\frac{x}{a}\to 0}\left(1+\frac{x}{a}\right)^{\frac{1}{\frac{x}{a}}}=e$$

19 (d)

20 (d)

Solution :

$$\int_{-\pi}^{\pi}(4+\pi\cos 2x)\,dx=\left[4x+\frac{1}{2}\pi\sin 2x\right]_{-\pi}^{\pi}$$

$$=[4\pi-(-4\pi)]=8\pi$$

21 (b)

Solution :

$y^2+2x^2=6$ (by differentiating both sides respect to x)

$$\therefore 2y\frac{dy}{dx}+4x=0$$

$$\therefore \frac{dy}{dx}=\frac{-2x}{y}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(1\,,\,2)}=\frac{-2}{2}=-1$$

measure of the positive angle which the tangent makes with positive direction of x-axis is 135°

22 (c)

Solution :

$$y=e^{ax} \qquad \therefore \frac{dy}{dx}=a\,e^{ax}$$

$$\therefore \frac{d^2y}{dx^2}=a^2e^{ax} \qquad \therefore \frac{d^3y}{dx^3}=a^3e^{ax}$$

$$\therefore \frac{d^4y}{dx^4}=a^4e^{ax}=a^4y$$

23 (c)

Solution :

$$\because \frac{dy}{dx}=\frac{\sqrt{2y+1}}{\sqrt{3x-2}} \qquad \therefore \frac{dy}{\sqrt{2y+1}}=\frac{dx}{\sqrt{3x-2}}$$

$$\therefore \int(2y+1)^{-\frac{1}{2}}dy=\int(3x-2)^{-\frac{1}{2}}dx$$

$$\therefore (2y+1)^{\frac{1}{2}}=\frac{2}{3}(3x-2)^{\frac{1}{2}}+c$$

$\therefore$ the curve passes through the point (1 , 4)

$$\therefore 3=\frac{2}{3}+c\Rightarrow c=\frac{7}{3}$$

Equation of the curve is ; $\sqrt{2y+1}=\frac{2}{3}\sqrt{3x-2}+\frac{7}{3}$

24 (a)

Solution :

$$\frac{d}{dx}[(\csc x-\cot x)(\csc x+\cot x)]$$

$$=\frac{d}{dx}(\csc^2 x-\cot^2 x)=\frac{d}{dx}(1)=\text{zero}$$

25 (b)

Solution :

$$\int\frac{dx}{\sqrt[3]{2x+9}}=\int(2x+9)^{\frac{-1}{3}}dx=\frac{3}{4}(2x+9)^{\frac{2}{3}}+c$$

Exam 18

1 (d)

Solution :

$$yx=1 \qquad \therefore y=x^{-1}$$

$$\therefore \frac{dy}{dx}=-x^{-2} \qquad \therefore \left(\frac{dy}{dx}\right)_{x=2}=-\frac{1}{4}$$

2 (d)

Solution :

$$x=e^{\theta}\cos\theta$$

$$\therefore \frac{dx}{d\theta}=e^{\theta}\cos\theta-e^{\theta}\sin\theta$$

$$,\ y=e^{\theta}\sin\theta$$

$$\therefore \frac{dy}{d\theta}=e^{\theta}\sin\theta+e^{\theta}\cos\theta$$

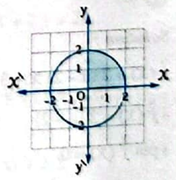
$$\therefore \frac{dy}{dx}=\frac{dy}{d\theta}\div\frac{dx}{d\theta}=\frac{e^{\theta}(\sin\theta+\cos\theta)}{e^{\theta}(\cos\theta-\sin\theta)}=\frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}$$

$\therefore \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}}$ is undefined.

$\therefore$ The tangent to the curve makes an angle of measure $\frac{\pi}{2}$ with the positive direction of x-axis.

3 (c)

Solution :



$$\int_0^2\sqrt{4-x^2}\,dx$$

Put $y=\sqrt{4-x^2}$

$$\therefore y^2=4-x^2$$

$$\therefore x^2+y^2=4$$

and this is the equation of a circle whose centre is the origin , its radius = 2

$\therefore y=\sqrt{4-x^2}$ is the part of circle curve above the x-axis

$\therefore$ The area $=\frac{1}{4}(\pi(2)^2)=\pi$ square unit.

4 (a)

Solution :

$$\because f(2x-1)=2x^3-4x^2+6x+7$$

(by differentiating both sides with respect to x)

$$2\acute{f}(2x-1)=6x^2-8x+6$$

at $2x-1=3 \quad \therefore x=2$

$$\therefore 2\acute{f}(3)=6\times 4-8\times 2+6$$

$$\therefore \acute{f}(3)=7$$

5 (b)

6 (d)

Solution :

$$\lim_{x\to\infty}\left(\frac{x+4}{x-2}\right)^{x+3}=\lim_{x\to\infty}\left(\frac{x-2+6}{x-2}\right)^{x+3}$$

$$=\lim_{x\to\infty}\left(1+\frac{6}{x-2}\right)^{x-2}\times\lim_{x\to\infty}\left(1+\frac{6}{x-2}\right)^{5}$$

$$=e^6\times 1^5=e^6$$

7 (d)

Solution :

$$\frac{1}{2}\int\frac{2x-1}{\sqrt{2x-1}}dx=\frac{1}{2}\int(2x-1)^{\frac{1}{2}}dx$$

$$=\frac{1}{6}(2x-1)^{\frac{3}{2}}+c$$

8 (b)

Solution :

$$f(x)=\begin{cases}x^2-2x & , \quad x\geq 2\\ -x^2+2x & , \quad x<2\end{cases}$$

$$\therefore \acute{f}(2^+)=\lim_{h\to 0^+}\frac{(2+h)^2-2(2+h)}{h}=2$$

$$\acute{f}(2^-)=\lim_{h\to 0^-}\frac{-(2+h)^2+2(2+h)}{h}=-2$$

$$\acute{f}(2^-)\neq\acute{f}(2^+)$$

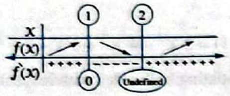
$\therefore$ The function is not differentiable.

At $x=2$

$$\therefore f'(x)=\begin{cases}2x-2 & , \quad x>2\\ \text{not exist} & , \quad x=2\\ -2x+2 & , \quad x<2\end{cases}$$

$\therefore f'(x)=0$ at $x=1$

undefined at $x=2$



$\therefore$ The function is increasing in $]-\infty , 1[$, $]2 , \infty[$

---

9 (c)

Solution :

$y=\sin x+\sqrt{3}\cos x \qquad \therefore \frac{dy}{dx}=\cos x-\sqrt{3}\sin x$

put $\frac{dy}{dx}=0$

$\therefore \cos x-\sqrt{3}\sin x=0$

$\therefore \tan x=\frac{1}{\sqrt{3}}$, $\frac{d^2y}{dx^2}=-\sin x-\sqrt{3}\cos x$

$\therefore x=\frac{\pi}{6}$, $\left(\frac{d^2y}{dx^2}\right)_{x=\frac{\pi}{6}}<0$

$\therefore$ y has maximum value at $x=\frac{\pi}{6}$

---

10 (a)

Solution :

$\because v=IR$ (by differentiation with respect to t)

$\therefore \frac{dv}{dt}=I\frac{dR}{dt}+R\frac{dI}{dt}$ (1)

, $\frac{dv}{dt}=1$ volt/sec. , $\frac{dI}{dt}=\frac{-1}{2}$ Ampere/sec.

at $v=12$ volt , $I=2$ Ampere

$\therefore R=\frac{v}{I}=\frac{12}{2}=6$ ohm.

By substitution in (1) :

$1=2\times\frac{dR}{dt}+6\times\frac{-1}{2} \qquad \therefore \frac{dR}{dt}=2$ ohm. /sec.

---

11 (c)

Solution :

$y=2\sin x-x\cos x$

$\therefore \frac{dy}{dx}=2\cos x-(x\times-\sin x+\cos x\times 1)$

$=2\cos x+x\sin x-\cos x$

$=\cos x+x\sin x$

$\therefore \frac{d^2y}{dx^2}=-\sin x+x\cos x+\sin x$

$=x\cos x$

$\therefore \frac{d^2y}{dx^2}+y=x\cos x+2\sin x-x\cos x=2\sin x$

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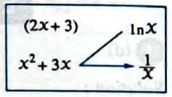
12 (d)

Solution :



$yz=(x^2+3x)\ln x$

$=x(x+3)\ln x$

---

13 (b)

Solution :

To fined points of intersection

$(2x^2)^2=4ax$

$\therefore 4x^4=4ax \qquad \therefore 4x^4-4ax=0$

$\therefore 4x(x^3-a)=0$

$\therefore x=0$ or $x=\sqrt[3]{a}$

$\int_0^{\sqrt[3]{a}}(y_2-y_1)\,dx=\int_0^{\sqrt[3]{a}}\left(\sqrt{4ax}-2x^2\right)dx$

$=\int_0^{\sqrt[3]{a}}\left(2a^{\frac{1}{2}}x^{\frac{1}{2}}-2x^2\right)dx$

$=\left[\frac{4}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}-\frac{2}{3}x^3\right]_0^{\sqrt[3]{a}}$

$=\frac{4}{3}a-\frac{2}{3}a=\frac{2}{3}a$

$\therefore \frac{2}{3}a=\frac{2}{3} \qquad \therefore a=1$

---

14 (b)

Solution :

$f'(x)=2(a-2)x+3$

, $f''(x)=2(a-2)$

, put $f''(x)=0 \qquad \therefore a=2$

at $a<2$, then $f''(x)<0$

i.e. The curve of the function f is convex upwards (concave downwords)

15 (a)

Solution :

$f(x)=2\sin\frac{x}{2}\cos\frac{x}{2}=\sin x$

$\therefore f'(x)=\cos x$

, $f''(x)=-\sin x$

, $f'''(x)=-\cos x$

$f^{(4)}(x)=\sin x$

$f^{(1000)}(x)=\sin x$

---

16 (d)

Solution :

$y_1=\frac{4}{x}$, $y_2=5-x$ intersected

at $x=1$

, $x=4$

$y_1\le y_2$ in this interval

$\therefore$ Volume $=\pi\int_1^4\left(y_2^2-y_1^2\right)dx$

$=\pi\int_1^4(5-x)^2-\left(\frac{4}{x}\right)^2dx$

$=\pi\int_1^4(25-10x+x^2-16x^{-2})\,dx$

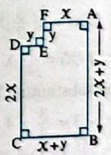
$=\pi\left[25x-5x^2+\frac{1}{3}x^3+16x^{-1}\right]_1^4$

$=9\pi$ cubic unit

---

17 (a)

Solution :



Let $CD=2AF=2x$

, $FE=ED=y$

$\therefore$ Perimeter of (ABCDEF)

$=6x+4y$

$\therefore 6x+4y=40 \qquad \therefore 3x+2y=20$

$\therefore y=10-\frac{3}{2}x$ (1)

$\therefore$ The area of (ABCDEF) $=xy+(x+y)(2x)$

$\therefore$ The area (A) $=3xy+2x^2$

$=3x(10-\frac{3}{2}x)+2x^2$

$=30x-\frac{9}{2}x^2+2x^2$

$=30x-\frac{5}{2}x^2$

$\therefore \frac{dA}{dx}=30-5x$

put $\frac{dA}{dx}=0 \qquad \therefore x=6$

From (1) : $\therefore y=1$, $\because \frac{d^2A}{dx^2}=-5<0$ (max.)

$\therefore$ The greatest area $=90$ cm^2.

---

18 (b)

Solution :

Let $f(x)=\frac{\cos x}{1+x^8}$

, $\because f(-x)=\frac{\cos(-x)}{1+(-x)^8}=\frac{\cos x}{1+x^8}=f(x)$

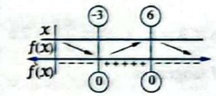
$\therefore$ The function is even.

$\therefore \int_{-\pi}^{\pi}\frac{2\cos x}{1+x^8}\,dx=4\int_0^{\pi}\frac{\cos x}{1+x^8}=4k$

---

19 (b)

Solution :



The curve has Maximum value at $x=6$

and Minimum value at $x=-3$

---

20 (a)

Solution :

Let the length of sides of the triangle are x , y and z

$\therefore x+y+z=40$ (by differentiate respect to time)

$\therefore \frac{dx}{dt}+\frac{dy}{dt}+\frac{dz}{dt}=0$

$\therefore \frac{dz}{dt}=7 \qquad \therefore \frac{dx}{dt}+\frac{dy}{dt}=-7$ (1)

, $\because x^2+y^2=z^2$ (by differentiate with respect to time)

$\therefore 2x\frac{dx}{dt}+2y\frac{dy}{dt}=2z\frac{dz}{dt}$

, $\because x=8$, $y=15$, $z=17$, $\frac{dz}{dt}=7$

, $2\times8\times\frac{dx}{dt}+2\times15\times\frac{dy}{dt}=2\times17\times7$

$\therefore 8\frac{dx}{dt}+15\frac{dy}{dt}=119$ (2)

by solving the two equations (1) , (2)

we get $\frac{dx}{dt}=-32$ cm./min. , $\frac{dy}{dt}=25$ cm./min.

21 (a)

Solution :

$y = \ln x$

$\therefore \grave{y} = \frac{1}{x} = x^{-1}$

, $\ddot{y} = -x^{-2}$

, $\dddot{y} = 2 \times 1 \times x^{-3}$

$y^{(4)} = -3 \times 2 \times 1 \times x^{-4}$

$\therefore y^{(10)} = -9 \times 8 \times 7 \times ... \times 1\, x^{-10} = \frac{-\lfloor 9}{x^{10}}$

---

22 (c)

---

23 (b)

Solution :

$\because y = x^2$, and let the point of tangency is (a , b)

$\therefore b = a^2$ (1)

$\therefore \frac{dy}{dx} = 2x$, at $x = a$, then $\frac{dy}{dx} = 2a$

$\therefore$ Equation of tangent is

$y - y_1 = m(x - x_1)$ $\quad y - b = 2a(x - a)$

$y - a^2 = 2a(x - a)$

$\therefore$ The point (3 , 5) lies on the tangent

$\therefore 5 - a^2 = 2a(3 - a)$ $\quad \therefore 5 - a^2 = 6a - 2a^2$

$\therefore a^2 - 6a + 5 = 0$ $\quad \therefore (a - 1)(a - 5) = 0$

a = 1 or a = 5 , then b = 1 or b = 25

$\therefore$ Equation of the tangent is $(y - 1) = 2(x - 1)$

i.e. $y = 2x - 1$

or $y - 25 = 10(x - 5)$

i.e. $y = 10x - 25$

---

24 (a)

Solution :

$\int x^5 \left(1 + \frac{3}{x}\right)^5 dx = \int (x + 3)^5 dx = \frac{1}{6}(x + 3)^6 + c$

---

25 (b)

Solution :

$\frac{dz}{dy} = \frac{dz}{dx} \div \frac{dy}{dx}$

$= \frac{2x - 3}{x^2 + 1}$

$\therefore \frac{d^2 z}{dy^2} = \frac{2(x^2 + 1) - (2x - 3) \times 2x}{(x^2 + 1)^2} \times \frac{1}{x^2 + 1}$

$= \frac{-2x^2 + 6x + 2}{(x^2 + 1)^3}$

$\therefore \left(\frac{d^2 z}{dy^2}\right)_{x=1} = \frac{-2 + 6 + 2}{2^3} = \frac{3}{4}$

Exam 19

1 (b)

Solution :

$x y^3 = 3$

$\therefore y = \sqrt[3]{\frac{3}{x}} = \sqrt[3]{3}\, x^{-\frac{1}{3}}$

$\therefore \frac{dy}{dx} = \frac{-\sqrt[3]{3}}{3} x^{-\frac{4}{3}}$

$\therefore \left(\frac{dy}{dx}\right)_{(3, 1)} = -\frac{1}{9}$

---

2 (d)

Solution :

Equation of the straight line $\overleftrightarrow{AB}$

$y = \frac{3}{2}x - 3$

$\therefore$ The first derivative of the function is

$\frac{3}{2}x - 3$

$\because$ The function f has local minimum value = − 3

$\therefore \frac{3}{2}x - 3 = 0 \quad \therefore x = 2$

$\therefore$ The curve passes through the point (2 , − 3)

, $f(x) = \int \left(\frac{3}{2}x - 3\right) dx$

$\therefore f(x) = \frac{3}{4}x^2 - 3x + c$

by substituting by $x = 2$, $f(x) = -3$

$\therefore -3 = \frac{3}{4} \times 4 - 6 + c \quad \therefore c = 0$

$\therefore \int_0^3 f(x)\, dx = \int_0^3 \left(\frac{3}{4}x^2 - 3x\right) dx$

$= \left(\frac{1}{4}x^3 - \frac{3}{2}x^2\right)_0^3$

$= \frac{-27}{4} - 0 = \frac{-27}{4}$

---

3 (d)

Solution :

$\because \frac{dy}{dx} = 2x + \frac{1}{2}\sec^2 \frac{x}{2}$

$dy = \int \left(2x + \frac{1}{2}\sec^2 \frac{x}{2}\right) dx$

$\therefore y = x^2 + \tan \frac{x}{2} + c$

, at $x = \frac{\pi}{2}$, $y = \frac{\pi^2}{4} + 9$

$\therefore \frac{\pi^2}{4} + 9 = \left(\frac{\pi}{2}\right)^2 + \tan\left(\frac{\pi}{4}\right) + c$

$\therefore c = 8$

$\therefore y = x^2 + \tan \frac{x}{2} + 8$

---

4 (a)

Solution :

$\int \frac{1 + \sin^2 x}{1 - \sin^2 x} dx = \int \frac{1 + \sin^2 x}{\cos^2 x} dx$

$= \int (\sec^2 x + \tan^2 x)\, dx$

$= \int (2\sec^2 x - 1)\, dx$

$= 2\tan x - x + c$

---

5 (a)

Solution :

$\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = \lim_{x \to 0} \frac{e^x (1 - e^{\sin x - x})}{x - \sin x}$

$\lim_{x \to 0} e^x \times \lim_{(\sin x - x) \to 0} \frac{e^{\sin x - x} - 1}{\sin x - x} = 1 \times 1 = 1$

---

6 (c)

Solution :

Let $f(x) = \frac{x}{x^4 + \cos x}$

$f(-x) = \frac{-x}{(-x)^4 + \cos(-x)} = \frac{-x}{(x)^4 + \cos x}$

$= -f(x)$

$\therefore f(x)$ is odd

$\therefore \int_{-a}^{a} \frac{x}{x^2 + \cos x} dx = \text{zero}$

---

7 (d)

Solution :

$2y = 3 - x^2$

(by differentiating both sides with respect to x)

$\therefore 2\frac{dy}{dx} = -2x$

$\therefore \frac{dy}{dx} = -x$

$\therefore \left(\frac{dy}{dx}\right)_{(1,1)} = -1$

$\therefore$ slope of the normal = 1

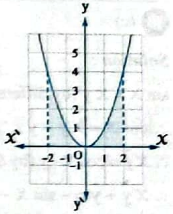
$\therefore$ equation of the normal is $\frac{y - 1}{x - 1} = 1$

i.e. $x - y = 0$

---

8 (c)

Solution :



volume $= \pi \int_{-2}^{2} y^2\, dx$

$= \pi \int_{-2}^{2} (x^2)^2\, dx$

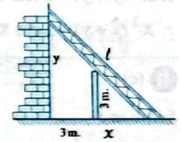
$= \frac{1}{5}\pi [x^5]_{-2}^{2}$

$= \frac{1}{5}\pi (32 - (-32)) = \frac{64}{5}\pi$

---

9 (d)

Solution :



Let the ladder length $= l$ metre.

, the height of the ladder top from the ground = y metre.

, the distance between the ladder base and the wall = x metre.

, from similarity :

$\frac{3}{y} = \frac{x}{x + 3}$ $\quad \therefore y = \frac{3x + 9}{x} = 3 + 9x^{-1}$

$\therefore \frac{dy}{dx} = \frac{-9}{x^2}$

, $\because l^2 = (x + 3)^2 + y^2$

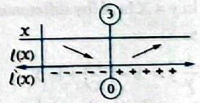
$\therefore 2l \frac{dl}{dx} = 2(x + 3) + 2y \frac{dy}{dx}$

$\therefore 2l \frac{dl}{dx} = 2(x + 3) + 2\left(3 + \frac{9}{x}\right)\left(\frac{-9}{x^2}\right)$

$= 2(x + 3) + 6(x + 3)\left(\frac{-9}{x^3}\right)$

$= 2(x + 3)\left[1 - \frac{27}{x^3}\right]$

, at the maximum or minimum values



, put $\frac{dl}{dx} = 0$

$\therefore x = -3$ (refused) or $\frac{27}{x^3} = 1$

$\therefore x = 3$

$\because$ the sign of $\frac{d\ell}{dx}$ changes from negative to positive around $x = 3$

$\therefore$ At $x = 3$, the length of the ladder is minimum.

$\therefore y = 3 + \frac{9}{3} = 6 \qquad \therefore \ell^2 = (3+3)^2 + 6^2 = 72$

$\therefore \ell = 6\sqrt{2}$ m.

---

10 (c)

Solution :

$\sin x = x\,y$ (by differentiation with respect to x

$\therefore \cos x = x\,y' + y$

$\therefore x\,y' = \cos x - y$ (by differentiation with respect to x)

$\therefore x\,y'' + y' = -\sin x - y'$, $x\,y'' + 2\,y' + \sin x = 0$

$\therefore x^2\,y'' + 2\,x\,y' + x \sin x = 0$

$\therefore x^2\,y'' + 2\,x\,y' + x^2\,y = 0$

$\therefore x^2\,(y'' + y) + 2\,(\cos x - y) = 0$

$\therefore x^2\,(y'' + y) + 2\cos x = 2\,y$

---

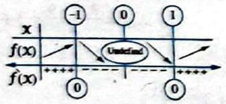
11 (c)

Solution :

$f(x) = x + \frac{1}{x} = x + x^{-1}$

$f'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2}$

, put $f'(x) = 0 \qquad \therefore x^2 = 1 \qquad \therefore x = \pm 1$



the local minimum value of the function at $x = 1$

---

12 (d)

Solution :

$y = x^x$ (by taking logarithm of both sides to the base e)

$\ln y = x \ln x$ (by differentiating both sides with respect to x)

$\frac{1}{y}\,y' = x \cdot \left(\frac{1}{x}\right) + \ln x$

$y' = y\,(\ln x + 1) = x^x\,(1 + \ln x)$

---

13 (a)

14 (c)

Solution :

$f(x) = 2\ln x - x^2$

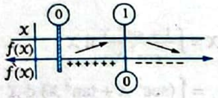
$\therefore$ Domain $=]0, \infty[$

$f'(x) = \frac{2}{x} - 2x = \frac{2 - 2x^2}{x}$

Put $f'(x) = $ zero , then $2 - 2x^2 = 0$

$\therefore x = 1$ or $x = -1 \notin$ Domain

$f'(x)$ undefined at $x = 0 \notin$ Domain



$\therefore$ The function is decreasing in $]1, \infty[$

---

15 (b)

Solution :

Dimensions of cuboid at any instant are $3 + 2t$, $4 + t$, $12 - 3t$

$\therefore$ Volume of the cuboid at any instant is $V = (3 + 2t)(4 + t)(12 - 3t)$

$\therefore \frac{dV}{dt} = (2)(4 + t)(12 - 3t) + (3 + 2t)(12 - 3t) + (3 + 2t)(4 + t)(-3)$

$\left(\frac{dV}{dt}\right)_{t=2} = (2)(6)(6) + (7)(6) + (7)(6)(-3)$

$= -12\ \text{cm}^3/\text{sec.}$

---

16 (b)

---

17 (b)

Solution :

Let the height of the mass = h and the distance of the car from the projection of the pulley = x

$\therefore BC = 12 - h$

, $AB = 25 - (12 - h) = 13 + h$

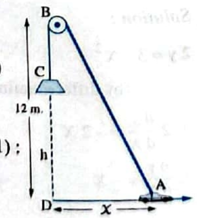
From the figure :

$\therefore (13 + h)^2 = x^2 + 12^2 \qquad (1)$

$\therefore 2\,(13 + h)\frac{dh}{dt} = 2\,x\,\frac{dx}{dt}$

at $x = 16$ m. , from equation (1) :

$\therefore (13 + h)^2 = 400$



$\therefore h = 7$ m.

$\therefore 2\,(13 + 7)\frac{dh}{dt} = 2 \times 16 \times 6 \qquad \therefore \frac{dh}{dt} = 4.8$ m./sec.

---

18 (b)

Solution :

$\because x = \sin y$ (by differentiating both sides with respect to x)

$\therefore 1 = \cos y\,\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$

---

19 (a)

Solution :

Slope of the normal $= (2y + 1)\csc x$

$\therefore$ Slope of the tangent $= \frac{dy}{dx} = \frac{-1}{(2y + 1)\csc x}$

(by integrating both sides after seperating variables respect to x)

$\int (2y + 1)\,dy = -\int \sin x\,dx$

$\therefore y^2 + y = \cos x + c$

$\because$ The curve passes through $(0, 0)$

$\therefore c = -1$

$\therefore$ The equation : $y^2 + y = \cos x - 1$

---

20 (d)

Solution :

Equation of the straight line $y_1 = -x + 3$

Equation of the curve : $x^2 = 4y$

$\therefore y_2 = \frac{1}{4}x^2$

$\because$ Equation of the curve

$y^2 = 4x$

$\therefore y_3 = 2\sqrt{x}$ (where y lies in the 1st quad.)

Points of intersection between the straight line (y_1) and the curve (y_2)

$\therefore -x + 3 = \frac{1}{4}x^2 \qquad \therefore x^2 + 4x - 12 = 0$

$\therefore (x + 6)(x - 2) = 0$

$\therefore x = -6 \notin$ 1st quad. , $x = 2 \in$ 1st quad.

Points of intersection between straight line (y_1) and the curve (y_3)

$\therefore -x + 3 = 2\sqrt{x}$

$x + 2\sqrt{x} - 3 = 0 \qquad \therefore (\sqrt{x} + 3)(\sqrt{x} - 1) = 0$

$\therefore \sqrt{x} = -3$ (refused)

or $x = 1$

$\therefore y = -1 + 3 = 2 \in$ 1st quad.

$\therefore$ The area $= {}_0\int^1 (y_3 - y_2)\,dx + {}_1\int^2 (y_1 - y_2)\,dx$

$= {}_0\int^1 \left(2x^{\frac{1}{2}} - \frac{1}{4}x^2\right)dx$

$+ {}_1\int^2 \left(-x + 3 - \frac{1}{4}x^2\right)dx$

$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{1}{12}x^3\right]_0^1 + \left[-\frac{1}{12}x^3 - \frac{1}{2}x^2 + 3x\right]_1^2$

$= \left(\frac{4}{3} - \frac{1}{12}\right) - (\text{zero}) + \left(\frac{-2}{3} - 2 + 6\right)$

$-\left(\frac{-1}{12} - \frac{1}{2} + 3\right) = \frac{13}{6}$ square unit.

---

21 (c)

Solution :

$\because A = (-4, 0)$, $B = (4, 0) \qquad \therefore C\,(4, f(4))$

$\therefore$ Slope of $\overline{AC} = f'(4) = \frac{f(4) - \text{zero}}{4 + 4}$

$\therefore 8\,f'(4) = f(4)$

$\because f(4) + f'(4) = 9$

$\therefore 8\,f'(4) + f'(4) = 9 \qquad \therefore 9\,f'(4) = 9$

$\therefore f'(4) = 1 \qquad \therefore f(4) = 8$

$\therefore$ Length of $\overline{BC} = 8$ length unit.

, $AB = 4 + 4 = 8$ length unit.

$\therefore$ Area of $\Delta ABC = \frac{1}{2} \times AB \times BC = \frac{1}{2} \times 8 \times 8$

$= 32$ square unit.

---

22 (a)

Solution :

$y = \ln(e^{2x} + e^x + 1)$

$\frac{dy}{dx} = \frac{2e^{2x} + e^x}{e^{2x} + e^x + 1}$

$\left(\frac{dy}{dx}\right)_{x=0} = 1$

, at $x = $ zero , $y = \ln(1 + 1 + 1) = \ln 3$

$\therefore$ The point is $(0, \ln 3)$

$\therefore$ Equation of the tangent is : $\frac{y - \ln 3}{x} = 1$

i.e. $y = x + \ln 3$

23 (c)

Solution :

$\because \tan(x^2 + y^2) = 0$

(by differentiating both sides with respect to x)

$\sec^2(x^2 + y^2) \times \left(2x + 2y\frac{dy}{dx}\right) = \text{zero}$

$\therefore 2x + 2y\frac{dy}{dx} = 0 \qquad \therefore \frac{dy}{dx} = \frac{-x}{y}$

---

24 (a)

Solution :

$\int_2^5 \frac{f'(x)}{f(x)}\,dx = [\ln|f(x)|]_2^5 = \ln|f(5)| - \ln|f(2)|$

$= \ln 6 - \ln 3 = \ln 2$

---

25 (d)

Solution :

$\because f(x) = x^{\sin x}$

(by taking logarithm of both sides to the base e)

$\ln\left(f(x)\right) = \sin x \ln x$

(by differentiating both sides with respect to x)

$\therefore \frac{f'(x)}{f(x)} = \sin x \times \frac{1}{x} + \ln x \times \cos x$

$\therefore f'(x) = x^{\sin x}\left(\frac{\sin x}{x} + \cos x \ln x\right)$

$\therefore f'\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\left(\frac{2}{\pi} + \text{zero}\right) = 1$

Exam 20

1 (b)

Solution :

Let $\sin x = y$

$\because f(\sin x) = \sin^2 x \qquad \therefore f(y) = y^2$

$\therefore f'(y) = 2y \qquad , f''(y) = 2 \qquad \therefore f''(1) = 2$

---

2 (c)

Solution :

$\because$ The point (a , b) satisfies the two equations of the straight line and the curve

$, \because \left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ (differentiating with respect to x)

$\therefore \frac{n}{a}\left(\frac{x}{a}\right)^{n-1} + \frac{n}{b}\left(\frac{y}{b}\right)^{n-1}\frac{dy}{dx} = 0$ at the point (a , b)

$\therefore \frac{n}{a} \times 1 + \frac{n}{b} \times 1 \times \frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{-n}{a} \times \frac{b}{n} = \frac{-b}{a}$ (slope of tangent at (a , b))

$, \because$ slope of straight line $= \frac{-1}{a} \div \frac{1}{b} = \frac{-b}{a}$

$\therefore$ The straight line touches the curve at the point (a , b) which lies on both of them for any value of n.

---

3 (b)

Solution :

$\because y = e^{2x}\cos x$

at $x = 0 \qquad \therefore y = 1$

$, \because \frac{dy}{dx} = -e^{2x}\sin x + 2e^{2x}\cos x$

$\left(\frac{dy}{dx}\right)_{x=0} = 2$

$\therefore$ Slope of the normal $= \frac{-1}{2}$

$\therefore$ The normal equation is $\frac{y-1}{x} = \frac{-1}{2}$

i.e. $2y + x = 2$

---

4 (b)

Solution :

Area of the circular sector (A) $= \frac{1}{2}r^2(x^{\text{rad}} - \sin x)$

$= 50(x^{\text{rad}} - \sin x)$

$\frac{dA}{dt} = 50\left(\frac{dx}{dt} - \cos x \cdot \frac{dx}{dt}\right) = 50\left(3 - \frac{1}{2} \times 3\right)$

$= 75$ cm²/min.

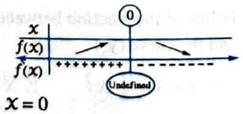
---

5 (b)

Solution :

$f(x) = 3 - \ln x^2$

$\therefore f'(x) = \frac{-2}{x}$



$\therefore f'(x)$ is undefined at $x = 0$

$\therefore$ The function is increasing in $]-\infty , 0[$

---

6 (a)

Solution :

$\lim\limits_{h \to 0} \frac{\cot(x+h) - \cot x}{h} = \frac{d}{dx}(\cot x) = -\csc^2 x$

7 (a)

Solution :

$\frac{dy}{dx} = a\csc^2 x$ (by integration with respect to x)

$\therefore y = -a\cot x + c$

$\therefore$ The curve passes through the points $\left(\frac{\pi}{4}, 5\right), \left(\frac{3\pi}{4}, 1\right)$

$\therefore 5 = -a + c$ (1)

$\therefore 1 = a + c$ (2)

$\therefore c = 3 \quad , \quad a = -2$

$\therefore$ The equation $y = 2\cot x + 3$

---

8 (d)

Solution :

Let radius length of the sector $= x$

and length of the arc $= y$

$\therefore$ Its perimeter $= 2x + y = 30$

$\therefore y = 30 - 2x$

$\therefore$ Area (A) $= \frac{1}{2}xy = \frac{1}{2}x(30 - 2x) = 15x - x^2$

$\therefore A' = 15 - 2x , A'' = -2$

$\therefore A' = 0$ at $x = 7.5$

$\because (A'')_{(x=7.5)} = -2 < 0$ (Max.)

$\therefore$ Radius length $= 7.5$ cm.

---

9 (c)

Solution :

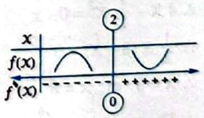
$y = x^3 - 6x^2$

$\therefore y' = 3x^2 - 12x$

$\therefore y'' = 6x - 12$

put $y'' = 0$

$\therefore x = 2$



$\therefore$ The curve is convex downwards in $]2 , \infty[$

---

10 (c)

Solution :

$\int \sec^{2017} x \tan x\,dx = \int \sec^{2016} x \sec x . \tan x\,dx$

$= \frac{1}{2017}\sec^{2017} x + c$

---

11 (c)

Solution :

$\because A_3 = \int_0^4 \frac{1}{4}x^2\,dx = \left[\frac{1}{12}x^3\right]_0^4 = \frac{16}{3}$ square units

$, A_2 = \int_0^4 \left(2\sqrt{x} - \frac{1}{4}x^2\right)dx = \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{1}{12}x^3\right]_0^4$

$= \frac{16}{3}$ square units

$\because$ area of the square $= 4 \times 4 = 16$ square units

$\therefore A_1 = 16 - \left(\frac{16}{3} + \frac{16}{3}\right) = \frac{16}{3}$ square units

$A_1 : A_2 : A_3 = \frac{16}{3} : \frac{16}{3} : \frac{16}{3} = 1 : 1 : 1$

---

12 (a)

Solution :

Let $y = x - \sin x \quad , \quad z = 1 - \cos x$

$\frac{dy}{dx} = 1 - \cos x \quad , \quad \frac{dz}{dx} = \sin x$

$\therefore \frac{dy}{dz} = \frac{dy}{dx} \div \frac{dz}{dx} = \frac{1 - \cos x}{\sin x}$

$\therefore \left(\frac{dy}{dz}\right)_{x=\frac{\pi}{3}} = \frac{\sqrt{3}}{3}$

---

13 (d)

---

14 (d)

Solution :

$y^2 = 18x$

$\therefore 2y\frac{dy}{dx} = 18$

$\therefore \frac{dy}{dx} = \frac{18}{2y} = \frac{9}{y}$

$\therefore \left(\frac{dy}{dx}\right)_{(2 , 6)} = \frac{9}{6} = \frac{3}{2}$

$\therefore$ The equation of the tangent is :

$\frac{y-6}{x-2} = \frac{3}{2}$

i.e. $3x - 2y + 6 = 0$

at $y = 0 \qquad \therefore x = -2$

, equation of the normal is :

$\frac{y-6}{x-2} = \frac{-2}{3}$

i.e. $2x + 3y - 22 = 0$

put $y = 0$, $x = 11$

$\therefore$ Length of $\overline{BC} = 11 + 2 = 13$ length units.

15 (c)

Solution :

Let the equation of the curve is :

$y = ax^3 + bx^2 + cx + d$

When a , b , c , d are real constant.

$\therefore y' = 3ax^2 + 2bx + c$

$y'' = 6ax + 2b$

$\because f''(x) < 0$ when $x < \frac{-2}{3}$

$f''(x) > 0$ when $x > \frac{-2}{3}$

$\therefore$ At $x = \frac{-2}{3}$ has point of inflection

$y'' = 0$, at $x = \frac{-2}{3}$

$\therefore -4a + 2b = 0$

$\therefore b = 2a$ (1)

$\because (-1 , 2)$ is critical point.

$\therefore y' = 0$, when $x = -1$

$\therefore 3a - 2b + c = 0$ (2)

$\because$ The curve passes through the point (1 , 6)

$\therefore a + b + c + d = 6$ (3)

$\because$ The curve passes through (– 1 , 2)

$-a + b - c + d = 2$ (4)

By solving the 4 equations , we get :

$\therefore a = 1 , b = 2 , c = 1 , d = 2$

$\therefore$ equation of the curve is :

$y = x^3 + 2x^2 + x + 2$

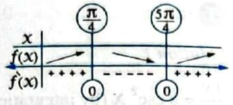
16 (c)

Solution :

$y = \sin x + \cos x$

$\therefore \frac{dy}{dx} = \cos x - \sin x$

put $\frac{dy}{dx}$ = zero



$\cos x = \sin x$

$\therefore x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$

at $x = \frac{\pi}{4}$ exists maximum value of the function and equal

$f\left(\frac{\pi}{4}\right) = \sqrt{2}$

17 (a)

Solution :

$h(x) = (f\circ g)(x) = f\big(g(x)\big) = f(\cot x) = \cot^2 x$

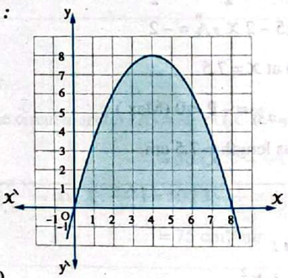
$\therefore h'(x) = 2\cot x \times -\csc^2 x = -2\cot x \csc^2 x$

$\therefore h'\left(\frac{\pi}{4}\right) = -2\cot\frac{\pi}{4}\csc^2\frac{\pi}{4} = -4$

18 (c)

19 (c)

Solution :



Put $y = 0$

$\therefore 4x - \frac{1}{2}x^2 = 0$

$\therefore \frac{1}{2}x(8 - x) = 0$

$\therefore x = 0$ or $x = 8$

$\therefore$ The area (A) $= \int_0^8 (4x - \frac{1}{2}x^2)\,dx$

$= \left[2x^2 - \frac{1}{6}x^3\right]_0^8$

$= \left[2(8)^2 - \frac{1}{6}(8)^3\right] - [\text{zero}]$

$= \frac{128}{3}$ square metre.

$\therefore$ The cost = the area × the cost of one square metre

$= \frac{128}{3} \times 1200 =$ L.E. 51200

20 (b)

Solution :

$\because \tan 60° = \frac{k}{3}$

$\therefore k = 3\sqrt{3}$

$\int_{-2}^{2} (f''(x) + f'(x))\,dx = [f'(x) + f(x)]_{-2}^{2}$

$= [f'(2) + f(2)]$

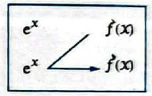
$- [f'(-2) + f(-2)]$

$= [-\sqrt{3} + 3\sqrt{3}]$

$- [\text{zero} + \text{zero}] = 2\sqrt{3}$

21 (c)

Solution :



$\int e^x (f'(x) + f''(x))\,dx$

$= \int e^x f'(x)\,dx + \int e^x f''(x)\,dx$

$= e^x f'(x) - \cancel{\int e^x f''(x)\,dx} + \cancel{\int e^x f''(x)\,dx}$

$= e^x f'(x) + c$

22 (d)

Solution :

$\because f(x) = \begin{cases} 5 - x^2 & , \quad -3 \le x \le 2 \\ x^2 - 3 & , \quad 2 < x \le 3 \end{cases}$

$f(2^-) = f(2^+) = f(2)$

$\therefore$ The function is continuous at $x = 2$

$\therefore f'(2^-) \ne f'(2^+)$

$\therefore f'(x) = \begin{cases} -2x & , \quad -3 \le x < 2 \\ \text{not exist} & , \quad x = 2 \\ 2x & , \quad 2 < x \le 3 \end{cases}$

In case $-3 \le x < 2$, put $f'(x) = 0$

$\therefore x = 0 \in [-3 , 2[$

$\therefore f(0) = 5$

In case $2 < x \le 3$, put $f'(x) = 0$

$\therefore x = 0 \notin]2 , 3]$

$\therefore f(2) = 1 , f(-3) = -4 , f(3) = 6$

$\therefore$ The function has an absolute minimum value $= -4$ at $x = -3$

23 (b)

Solution :

$\because f$ is a polynomial function of fifth degree.

$\therefore f''$ is a polynomial function of third degree

$\therefore$ The greatest number of roots to the function = 3 and at these points there may be inflection points

24 (a)

Solution :

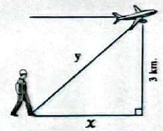
The volume $= \pi \int_1^e y^2\,dx = \pi \int_1^e \left(\frac{\ln x}{\sqrt{x}}\right)^2 dx$

$= \pi \int_1^e \frac{1}{x}(\ln x)^2\,dx = \pi\left[\frac{1}{3}(\ln x)^3\right]_1^e$

$= \pi\left(\frac{1}{3} - 0\right) = \frac{1}{3}\pi$ cube units.

25 (b)

Solution :



$\frac{dx}{dt} = 480$ km/h.

$\therefore 9 + x^2 = y^2$

$\therefore 2x\frac{dx}{dt} = 2y\frac{dy}{dt}$

$\therefore x\frac{dx}{dt} = y\frac{dy}{dt}$

After 30 sec.

$\therefore t = \frac{30}{60 \times 60} = \frac{1}{120}$ h.

$x = 480 \times \frac{1}{120} = 4$ km. , $y = \sqrt{3^2 + 4^2} = 5$ km.

$\therefore 4 \times 480 = 5 \times \frac{dy}{dt}$

$\therefore \frac{dy}{dt} = 384$ km./h.

Answers

of

School Book Examinations

Answers of school book examinations

Model 1

1

(1) (c) (2) (b) (3) (b)

(4) (b) (5) (d) (6) (c)

2

[a] $\int \sin x \cos^3 x \, dx = -\int -\sin x \cos^3 x \, dx$

$= -\frac{1}{4} \cos^4 x + c$

[b] $e^{xy} - x^2 + y^3 = 0$ (1)

at $x = 0$ $\therefore 1 - 0 + y^3 = 0$

$\therefore y^3 = -1$ $\therefore y = -1$

(By differentiating (1) with respect to x)

$\therefore e^{xy} (y + x y`) - 2x + 3y^2 y` = 0$, at $(0 , -1)$

$\therefore 1(-1 + 0) - 0 + 3(-1)^2 y` = 0$

$\therefore -1 + 3y` = 0$ $\therefore y` = \frac{1}{3}$ $\therefore \left(\frac{dy}{dx}\right)_{x=0} = \frac{1}{3}$

3

[a] $x^2 - 3xy - y^2 + 3 = 0$

(By differentiating with respect to x)

$\therefore 2x - 3y - 3x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{2x - 3y}{3x + 2y}$ $\therefore \left(\frac{dy}{dx}\right)_{(-1 , 4)} = \frac{-14}{5}$

$\therefore$ The equation of the tangent is : $\frac{y - 4}{x + 1} = \frac{-14}{5}$

$\therefore 14x + 5y - 6 = 0$

[b] Let the two right sides after t minutes are :

$\left(6 + \frac{1}{3}t\right)$, $(30 - t)$

(1) Area of triangle A $= \frac{1}{2}\left(6 + \frac{1}{3}t\right) \times (30 - t)$

$= \frac{1}{2}\left(180 + 4t - \frac{1}{3}t^2\right)$

$= 90 + 2t - \frac{1}{6}t^2$

$\therefore \frac{dA}{dt} = 2 - \frac{1}{3}t$

at $t = 3$: $\frac{dA}{dt} = 2 - \frac{1}{3}(3) = 1$ cm²/min.

(2) At $\frac{dA}{dt} = 0$ $\therefore 2 - \frac{1}{3}t = 0$

$\therefore t = 6$

4

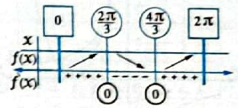
[a] $f(x) = x + 2\sin x$ $\because$ The domain $=]0 , 2\pi[$

, $f`(x) = 1 + 2\cos x$

, putting : $f`(x) = 0$ $\therefore 1 + 2\cos x = 0$

$\therefore \cos x = \frac{-1}{2}$ $\therefore x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

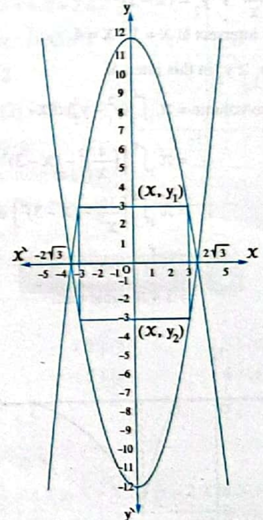
$\therefore$ The critical point is at $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$



$\therefore f$ is increasing on $]0 , \frac{2\pi}{3}[$, $]\frac{4\pi}{3} , 2\pi[$

and decreasing on $]\frac{2\pi}{3} , \frac{4\pi}{3}[$

[b]



$y_1 = 12 - x^2$, $0 < x < 2\sqrt{3}$

$y_2 = x^2 - 12$, $0 < x < 2\sqrt{3}$

$\therefore$ The dimensions of rectangle $= 2x$ horizontal

, vertical $= y_1 - y_2 = (12 - x^2) - (x^2 - 12)$

$= 12 - x^2 + 12 - x^2 = 24 - 2x^2$

$\therefore$ Area of rectangle (A) $= 2x(24 - 2x^2)$

$= 48x - 4x^3$

$\therefore A` = 48 - 12x^2$

Putting : $\grave{A} = 0$ $\quad \therefore x^2 = 4$ $\quad \therefore x = 2$

, $\grave{\grave{A}} = -24\,x$ $\quad \therefore [\grave{\grave{A}}]_{x=2} < 0$

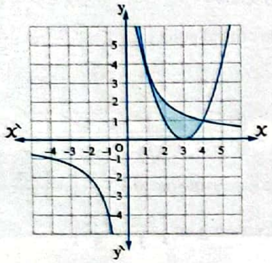
$\therefore$ At $x = 2$ A has a maximum value.

$\therefore$ The greatest area of rectangle

$A = 48 \times 2 - 4 \times 2^3 = 64$

5

[a]



$y_1 = \frac{4}{x}$, $y_2 = (x-3)^2$

They intersect at $x = 1$, $x = 4$

, $\because y_1 \geq y_2$ in this interval

$\therefore$ The volume $= \pi \int_1^4 (y_1^2 - y_2^2)\,dx$

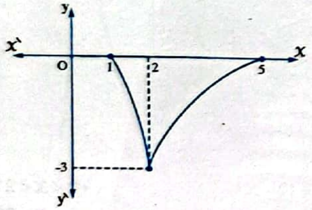
$= \pi \int_1^4 \left[\left(\frac{4}{x}\right)^2 - (x-3)^4\right] dx$

$= \pi \int_1^4 \left[\frac{16}{x^2} - (x-3)^4\right] dx$

$= \pi \left[-16\,x^{-1} - \frac{1}{5}(x-3)^5\right]_1^4$

$= 5.4\,\pi$ cubic unit.

[b]



Model 2

1

(1) (b)	(2) (c)	(3) (b)
(4) (a)	(5) (c)	(6) (c)

2

[a] (1) $\int x(2x-1)^3\,dx$

Putting $z = 2x - 1$

$\therefore x = \frac{1}{2}z + \frac{1}{2}$ $\qquad \therefore dx = \frac{1}{2}\,dz$

$\therefore \int x(2x-1)^3\,dx$

$= \int \left(\frac{1}{2}z + \frac{1}{2}\right).z^3.\frac{1}{2}\,dz$

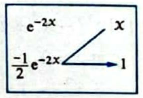
$= \int \left(\frac{1}{4}z^4 + \frac{1}{4}z^3\right) dz = \frac{1}{20}z^5 + \frac{1}{16}z^4 + c$

$= \frac{1}{20}(2x-1)^5 + \frac{1}{16}(2x-1)^4 + c$

(2) $\int x e^{-2x}\,dx$

$= \frac{-1}{2}x e^{-2x} + \int \frac{1}{2} e^{-2x}\,dx$

$= \frac{-1}{2}x e^{-2x} - \frac{1}{4} e^{-2x} + c$



[b] Let $y = \sqrt{16 + x^2}$, $z = \frac{x}{x-2}$

$\therefore \frac{dy}{dx} = \frac{2x}{2\sqrt{16+x^2}} = \frac{x}{\sqrt{16+x^2}}$

, $\frac{dz}{dx} = \frac{(x-2) - x}{(x-2)^2} = \frac{-2}{(x-2)^2}$

$\therefore \frac{dy}{dz} = \frac{dy}{dx} \div \frac{dz}{dx} = \frac{x}{\sqrt{16+x^2}} \div \frac{-2}{(x-2)^2}$

$= \frac{-x(x-2)^2}{2\sqrt{16+x^2}}$

At $x = -3$: $\frac{dy}{dz} = \frac{-(-3)(-3-2)^2}{2\sqrt{16+(-3)^2}} = \frac{75}{10} = \frac{15}{2}$

3

[a] $x \cos y + y \cos x = 1$

(By differentiating with respect to x)

$\therefore \cos y + x(-\sin y)\frac{dy}{dx} + \frac{dy}{dx}\cos x$

$+ y(-\sin x) = 0$

$\therefore \frac{dy}{dx}(-x \sin y + \cos x) = -\cos y + y \sin x$

$\therefore \frac{dy}{dx} = \frac{y \sin x - \cos y}{\cos x - x \sin y}$

[b] $f(x) = 2x^3 + 6x^2 + 5$ $\qquad \therefore f`(x) = 6x^2 + 12x$

Putting $f`(x) = 0 : 6x^2 + 12x = 0$

$\therefore 6x(x+2) = 0$ $\qquad \therefore x = 0 \in [-1, 1]$

or $x = -2 \notin [-1, 1]$

$f(-1) = 2(-1)^3 + 6(-1)^2 + 5 = 9$

, $f(0) = 2(0)^3 + 6(0)^2 + 5 = 5$

, $f(1) = 2(1)^3 + 6(1)^2 + 5 = 13$

The function has absolute maximum value = 13 at $x = 1$ and absolute minimum value = 5 at $x = 0$

4

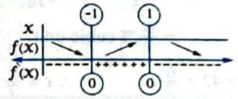
[a] (1) $f(x) = \begin{cases} 2x + x^2 & , \quad x < 0 \\ 2x - x^2 & , \quad x \geq 0 \end{cases}$

$\therefore f`(x) = \begin{cases} 2 + 2x & , \quad x < 0 \\ 2 - 2x & , \quad x > 0 \end{cases}$

$f`(0^+) = f`(0^-)$

$f`(x) = 0 : 2 + 2x = 0$ $\quad \therefore x = -1 < 0$

, $2 - 2x = 0$ $\qquad \therefore x = 1 > 0$



The function has a local minimum value at $x = -1$

, $f(-1) = 2(-1) + (-1)^2 = -1$

and has a local maximum value at $x = 1$

, $f(1) = 2(1) - (1)^2 = 1$

(2) $\int_{-1}^{3} f(x)\,dx = \int_{-1}^{0} (2x + x^2)\,dx$

$+ \int_0^3 (2x - x^2)\,dx = \left[x^2 + \frac{1}{3}x^3\right]_{-1}^{0}$

$+ \left[x^2 - \frac{1}{3}x^3\right]_0^3 = \left[(0) - \left((-1)^2 + \frac{1}{3}(-1)^3\right)\right]$

$+ \left[\left((3)^2 - \frac{1}{3}(3)^3\right) - (0)\right] = \frac{-2}{3}$

[b] Volume of cube $v = x^3$

(By differentiating with respect to time)

$\therefore \frac{dv}{dt} = 3x^2 \frac{dx}{dt}$

, at $\frac{dv}{dt} = 27$ cm³/min. , $x = 3$

$\therefore 27 = 3(3)^2 . \frac{dx}{dt}$ $\qquad \therefore \frac{dx}{dt} = 1$

, area of all face $A = 6x^2$

(By differentiating with respect to time)

$\therefore \frac{dA}{dt} = 12x . \frac{dx}{dt}$, at $x = 3$, $\frac{dx}{dt} = 1$

$\therefore \frac{dA}{dt} = 12(3).(1) = 36$ cm²/min.

5

[a] The points of intersection $x^2 = 6x - x^2$

$2x^2 - 6x = 0$

$2x(x-3) = 0$ $\qquad \therefore x = 0$ or $x = 3$

$\therefore$ The area $= \int_0^3 \left[(6x - x^2) - (x^2)\right] dx$

$= \int_0^3 (6x - 2x^2)\,dx$

$= \left[3x^2 - \frac{2}{3}x^3\right]_0^3$

$= [27 - 18] - [0] = 9$ square units.

[b] $f(x) = x^3 + a x^2 + b x$

Putting : $x = 2$, $y = 2$

$\therefore 2 = (2)^3 + a(2)^2 + b(2)$ $\qquad \therefore 2 = 8 + 4a + 2b$

$\therefore 2a + b = -3$ (1)

, $f`(x) = 3x^2 + 2ax + b$

, $f``(x) = 6x + 2a$

Putting : $f``(x) = 0$

, $x = 2$

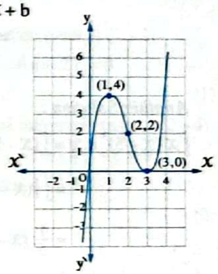
$0 = 6(2) + (2a)$

$\therefore a = -6$

Substituting in (1) :

$\therefore 2(-6) + b = -3$

$\therefore b = 9$



Model 3

1

(1) (b)	(2) (b)	(3) (d)
(4) (c)	(5) (a)	(6) (a)

2

[a] $y = x^2 \ln x$

$\therefore \frac{dy}{dx} = 2x \ln x + x^2 \times \frac{1}{x} = 2x \ln x + x$

$= x(2 \ln x + 1)$

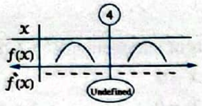
[b] $f(x) = \sqrt[3]{(x-4)^2} = (x-4)^{\frac{2}{3}}$

$\therefore f`(x) = \frac{2}{3}(x-4)^{\frac{-1}{3}}$

, $f`(4)$ undefined because it has a vertical tangent

, $f``(x) = \frac{-2}{9}(x-4)^{\frac{-4}{3}} = \frac{-2}{9\sqrt[3]{(x-4)^4}}$

, $f``(x)$ undefined at $x = 4$



The curve is convex upwards at $]-\infty, 4[$, $]4, \infty[$ and has no inflection points.

3

[a] (1) $\int x(x-5)^3\, dx$

Putting : $z = x - 5$ $\therefore x = z + 5$

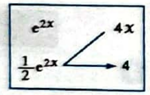
$dx = dz$

$$\therefore \int x(x-5)^3\, dx = \int (z+5)(z)^3\, dz$$
$$= \int (z^4 + 5z^3)\, dz$$
$$= \frac{1}{5}z^5 + \frac{5}{4}z^4 + c$$
$$= \frac{1}{5}(x-5)^5 + \frac{5}{4}(x-5)^4 + c$$

Another Solution :

$$\int x(x-5)^3\, dx = \int (x-5+5)(x-5)^3\, dx$$
$$= \int [(x-5)^4 + 5(x-5)^3]\, dx$$
$$= \frac{1}{5}(x-5)^5 + \frac{5}{4}(x-5)^4 + c$$

(2) $\int 4x e^{2x}\, dx$



$$= 2x e^{2x} - \int 2e^{2x}\, dx$$
$$= 2x e^{2x} - \frac{2e^{2x}}{2} + c$$
$$= 2x e^{2x} - e^{2x} + c$$

[b] $f(x) = x^4 - 4x^3$

(By differentiating with respect to x)

$\therefore f'(x) = 4x^3 - 12x^2$

, $f'(x) = 0 : 4x^3 - 12x^2 = 0$

$\therefore 4x^2(x-3) = 0$

$\therefore x = 0$ or $x = 3$

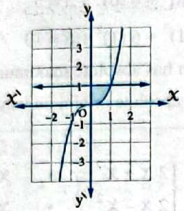
, $f(0) = 0$, $f(3) = -27$, $f(4) = 0$

f has absolute maximum value $= 0$

at $x = 0$, $x = 4$

4

[a]



$y_1 = 1$, $y_2 = x^3$

, $\because y_1 \geq y_2$ on $[0, 1]$

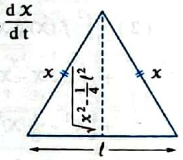
$$\therefore \text{The volume} = \pi \int_0^1 (y_1^2 - y_2^2)\, dx$$
$$= \pi \int_0^1 (1 - x^6)\, dx$$
$$= \pi \left[x - \frac{1}{7}x^7\right]_0^1$$
$$= \frac{6}{7}\pi \text{ cubic unit.}$$

$\therefore \pi r^2 \times 42 = \frac{6}{7}\pi$ $\therefore r = \frac{1}{7}$ length unit.

[b] Let the two equal side be x and the base ℓ

$$\therefore A = \frac{1}{2}\ell \times \sqrt{x^2 - \frac{1}{4}\ell^2}$$

(By differentiating with respect to time)



$$\frac{dA}{dt} = \frac{1}{2}\ell \times \frac{2x}{2\sqrt{x^2 - \frac{1}{4}\ell^2}} \cdot \frac{dx}{dt}$$
$$= \frac{\ell x}{2\sqrt{x^2 - \frac{1}{4}\ell^2}} \cdot \frac{dx}{dt}$$

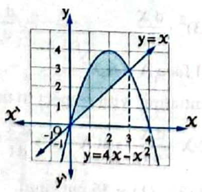
putting $\frac{dx}{dt} = -3$ cm./min. , $x = \ell$

$$\therefore \frac{dA}{dt} = \frac{\ell^2}{2\sqrt{\ell^2 - \frac{1}{4}\ell^2}}(-3) = -\sqrt{3}\,\ell$$

$\therefore$ The rate of decrease $= \sqrt{3}\,\ell$ cm.²/min.

5

[a]



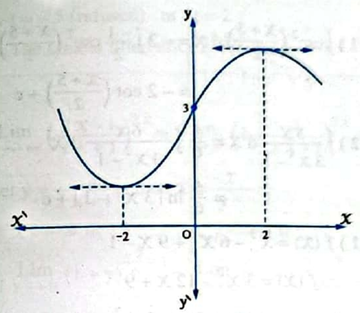
$\because x - y = 0$, $y = x$

The points of intersection : $x = 4x - x^2$

$\therefore 3x - x^2 = 0$ $\therefore x(3 - x) = 0$ $\therefore x = 0$ or $x = 3$

$$\text{Area } a = \int_0^3 [(4x - x^2) - (x)]\, dx$$
$$= \int_0^3 (3x - x^2)\, dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3$$
$$= \left[\frac{3}{2}(3)^2 - \frac{1}{3}(3)^3\right] - [0] = \frac{9}{2} \text{ square unit.}$$

[b]



Model 4

1

(1) (b)	(2) (b)	(3) (c)
(4) (d)	(5) (c)	(6) (c)

2

[a] (1) $\int (3x^2 - 4e^{2x})\, dx = x^3 - \frac{4e^{2x}}{2} + c$

$$= x^3 - 2e^{2x} + c$$

(2) $\int \frac{x-1}{\sqrt{x+3}}\, dx$

Let $z = x + 3$ $\therefore x = z - 3$

, $dx = dz$, $x - 1 = z - 4$

$$\therefore \text{The integration} = \int \frac{z-4}{\sqrt{z}}\, dz$$
$$= \int \left(z^{\frac{1}{2}} - 4z^{\frac{-1}{2}}\right) dz = \frac{2}{3}z^{\frac{3}{2}} - 8z^{\frac{1}{2}} + c$$
$$= \frac{2}{3}z^{\frac{1}{2}}[z - 12] + c$$
$$= \frac{2}{3}\sqrt{x+3}\,[x + 3 - 12] + c$$
$$= \frac{2}{3}\sqrt{x+3}\,[x - 9] + c$$

[b] $\sin y + \cos 2x = 0$

(By differentiating with respect to x)

$$\cos y \frac{dy}{dx} - 2\sin 2x = 0$$

(By differentiating with respect to x another time)

$$\therefore \cos y \frac{d^2y}{dx^2} + \frac{dy}{dx} \times -\sin y \times \frac{dy}{dx} - 4\cos 2x = 0 \text{ (Dividing by } \cos y)$$

$$\therefore \frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 \tan y = 4\cos 2x \sec y$$

3

[a] $\int_1^4 [f(x) + 2g(x) - 4]\, dx$

$$= \int_1^4 f(x)\, dx + 2\int_1^4 g(x)\, dx - 4\int_1^4 dx$$
$$= 7 + 2 \times -3 - 4[x]_1^4 = 7 + (-6) - 4[4 - 1] = -11$$

[b] $f(x) = ax^3 + bx^2 + cx + d$

$\therefore f'(x) = 3ax^2 + 2bx + c$

, $f''(x) = 6ax + 2b$

$\because$ The curve has inflection point at $(1, 2)$

$\therefore f''(1) = 0$ $\therefore 6a + 2b = 0$

$\therefore b = -3a$ (1)

$\because$ The curve has a local maximum value at $(2, 4)$

$\therefore f'(2) = 0$ $\therefore 12a + 4b + c = 0$

From (1) : $\therefore 12a - 12a + c = 0$

$\therefore c = 0$ (2)

$\because$ The curve passes through $(1, 2)$

$\therefore f(1) = 2$ $\therefore a + b + c + d = 2$

From (1) and (2) : $\therefore a - 3a + d = 2$

$\therefore -2a + d = 2$ (3)

$\because$ The curve passes through $(2, 4)$

$\therefore f(2) = 4$ $\therefore 8a + 4b + 2c + d = 4$

From (1) , (2) : $\therefore 8a - 12a + 0 + d = 4$

$\therefore -4a + d = 4$ (4)

By solving the two equations (3) and (4) :

$\therefore a = -1$, $d = 0$ $\therefore b = 3$

$\therefore$ The equation of the curve is :

$f(x) = -x^3 + 3x^2$

4

[a] $\sqrt{x} + \sqrt{y} = 1$ $\therefore \sqrt{y} = 1 - \sqrt{x}$

$\therefore y = 1 - 2\sqrt{x} + x$

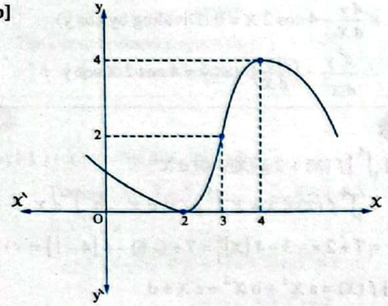
, at $x = 0$ $\therefore y = 1$

, at $y = 0$ $\therefore x = 1$

$\therefore$ Area $= {}_0\int^1 (1 - 2x^{\frac{1}{2}} + x)\, dx$

$= \left[x - \frac{4}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2\right]_0^1 = \frac{1}{6}$ square unit.

[b]



5

[a] The point of intersection

$y_1 = \frac{4}{x}$, $y_2 = 5 - x$

$\frac{4}{x} = 5 - x$ $\qquad \therefore 4 = 5x - x^2$

$\therefore x^2 - 5x + 4 = 0$ $\qquad \therefore x = 1$ or $x = 4$

$\therefore v = \pi\, {}_1\int^4 (y_2^2 - y_1^2)\, dx$

$= \pi\, {}_1\int^4 \left((5-x)^2 - (4x^{-1})^2\right) dx$

$= \pi\, {}_1\int^4 \left((25 - 10x + x^2) - (16x^{-2})\right) dx$

$= \pi\left[25x - 5x^2 + \frac{1}{3}x^3 + 16x^{-1}\right]_1^4$

$= \pi\left[(100 - 80 + \frac{1}{3}(64) + 4)\right.$
$\left.- (25 - 5 + \frac{1}{3} + 16)\right] = 9\pi$ cubic unit.

[b] $A = \pi(r_2^2 - r_1^2)$

(By differentiating with respect to time)

$\therefore \frac{dA}{dt} = \pi\left(2r_2\frac{dr_2}{dt} - 2r_1\frac{dr_1}{dt}\right)$

, $r_1 = 6$ cm. , $\frac{dr_1}{dt} = 0.3$ cm./sec.

, $r_2 = 10$ cm. , $\frac{dr_2}{dt} = -0.2$ cm./sec.

$\therefore \frac{dA}{dt} = \pi\left(2(10)(-0.2) - 2(6)(0.3)\right) = -7.6\pi$

$\therefore$ The rate of decrease of area (A) included between the two circles $= 7.6\pi$ cm²/sec.

Model 5

1

(1) $]-\infty, 0[\cup]2, \infty[$ $\qquad$ (2) $\{0, 2\}$

(3) $]1, \infty[$ $\qquad$ (4) 0

(5) 2 $\qquad$ (6) 4

2

[a] (1) $\int \csc^2\left(\frac{x+5}{2}\right) dx = -2\int \frac{-1}{2}\csc^2\left(\frac{x+5}{2}\right) dx$

$= -2\cot\left(\frac{x+5}{2}\right) + c$

(2) $\int \frac{5x}{3x^2-1}\, dx = \frac{5}{6}\int \frac{6x}{3x^2-1}\, dx$

$= \frac{5}{6}\ln|3x^2 - 1| + c$

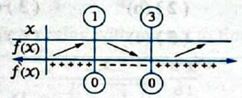
[b] (1) $f(x) = x^3 - 6x^2 + 9x - 1$

$\therefore f'(x) = 3x^2 - 12x + 9$

Putting : $f'(x) = 0$: $\therefore 3x^2 - 12x + 9 = 0$

$\therefore 3(x^2 - 4x + 3) = 0$

$\therefore 3(x-3)(x-1) = 0$ $\qquad \therefore x = 3$, $x = 1$



f is increasing on $]-\infty, 1[$, $]3, \infty[$

and decreasing on $]1, 3[$

(2) $f(0) = -1$, $f(1) = 3$, $f(2) = 1$

$\therefore f$ has absolute maximum value $= 3$ at $x = 1$

3

[a] $y = 4 + \cot x - \sec^2 x$

$\therefore \frac{dy}{dx} = -\csc^2 x - 2\sec x \cdot \sec x \tan x$

$= -\csc^2 x - 2\sec^2 x \tan x$

$\therefore \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = -6$

$\therefore$ The slope of tangent $= -6$

$\therefore$ The slope of normal $= \frac{1}{6}$

$\because$ At $x = \frac{\pi}{4}$ $\quad \therefore y = 3$ $\quad \therefore$ The point $\left(\frac{\pi}{4}, 3\right)$

$\therefore$ The equation of normal is : $(y - 3) = \frac{1}{6}\left(x - \frac{\pi}{4}\right)$

$\therefore 4x - 24y - \pi + 72 = 0$

[b] $\frac{dv}{dt} = 2t + 3$ (By integration)

$\therefore \int dv = \int (2t + 3)\, dt$ $\qquad \therefore v = t^2 + 3t + c$

at $t = 0$, $v = 0$ (Tank is empty)

$\therefore c = 0$ $\qquad \therefore v = t^2 + 3t$

, at $v = 10$: $10 = t^2 + 3t$

$\therefore t^2 + 3t - 10 = 0$ $\qquad \therefore (t+5)(t-2) = 0$

$\therefore t = -5$ (refused) or $t = 2$

$\therefore$ The tank is filled after 2 minutes.

4

[a] $\lim_{x\to\infty}\left(\frac{2x-1}{2x+1}\right)^{2x} = \lim_{x\to\infty}\left(1 - \frac{2}{2x+1}\right)^{2x}$

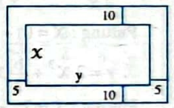
Let $y = \frac{-2}{2x+1}$, thus $x = \frac{-1}{y} - \frac{1}{2}$

$\because x \longrightarrow \infty$ $\qquad \therefore y \longrightarrow 0$

$\therefore \lim_{y\to 0}(1+y)^{\frac{-2}{y} - 1} = \lim_{y\to 0}\left((1+y)^{\frac{1}{y}}\right)^{-2} \times (1+y)^{-1}$

$\therefore e^{-2} \times 1^{-1} = \frac{1}{e^2}$

[b] Area of sticker $= 800 = xy$



$\therefore y = \frac{800}{x}$

Area of rectangle

$= (x + 20)(y + 10)$

$\therefore A = xy + 10x + 20y + 200$

$= 800 + 10x + 20 \times \frac{800}{x} + 200$

$\therefore A' = 10 - 16000x^{-2}$

, $A'' = 32000x^{-3}$

Putting : $A' = 0$ $\quad \therefore x = 40$ $\quad$, $\because A''_{x=40} > 0$

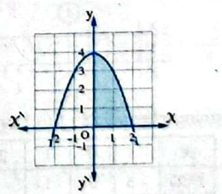
$\therefore x = 40$ makes the area as small as possible.

$\therefore$ Dimensions of rectangle $= 40 + 20 = 60$ cm.

, $\frac{800}{40} + 10 = 30$ cm..

5

[a]



$y = 4 - x^2$ $\qquad \because$ Rotation about the x-axis

$\therefore$ The volume $= \pi\, {}_0\int^2 y^2\, dx$

$= \pi\, {}_0\int^2 (4 - x^2)^2\, dx$

$= \pi\, {}_0\int^2 (16 - 8x^2 + x^4)\, dx$

$= \pi\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2$

$= \frac{256}{15}\pi$ cubic unit.

[b] $f(x) = x^3 + ax^2 + bx + 4$

$\therefore f'(x) = 3x^2 + 2ax + b$

, $f''(x) = 6x + 2a$ $\qquad$, $\because f'(x) = 0$ at $x = 2$

$\therefore 0 = 3(2)^2 + 2a(2) + b$

$\therefore 4a + b = -12$ $\qquad$ (1)

, $f''(x) = 0$ at $x = 1$

$\therefore 0 = 6(1) + 2a$ $\qquad \therefore a = -3$ $\qquad$ (2)

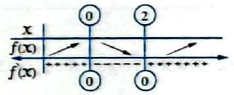
From (1) , (2) : $\therefore b = 0$

$\therefore f(x) = x^3 - 3x^2 + 4$

, putting $f'(x) = 0$

$\therefore 3x^2 - 6x = 0$

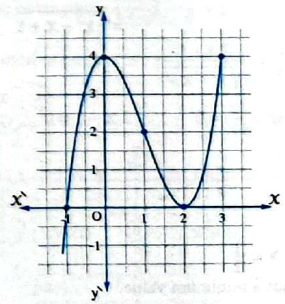
$\therefore x = 0$ or $x = 2$



* We find some assistance points :

$f(-1) = 0$, $f(3) = 4$

x	−1	0	1	2	3
$f(x)$	0	4	2	0	4



Model 6

1

(1) (b) (2) (a) (3) (a)
(4) (b) (5) (a) (6) (b)

2

[a] (1) $\int \frac{7x^3}{2-5x^4}\,dx = \frac{-7}{20}\int \frac{-20x^3}{2-5x^4}\,dx$

$= \frac{-7}{20}\ln|2-5x^4| + c$

(2) $\int\left(3e^{-5x}+\frac{\pi}{x}\right)dx = \frac{-3}{5}e^{-5x}+\pi\ln|x|+c$

[b] $y = a e^{x^2+1}$ $\therefore \frac{dy}{dx} = 2ax e^{x^2+1}$

$, \frac{d^2y}{dx^2} = 2ax \cdot 2x e^{x^2+1} + 2a e^{x^2+1}$

$= 4ax^2 e^{x^2+1} + 2a e^{x^2+1}$

$= 2a e^{x^2+1}[2x^2+1]$

$, \frac{d^3y}{dx^3} = 2a e^{x^2+1}[4x] + 4ax e^{x^2+1}[2x^2+1]$

$= 2a e^{x^2+1}[4x+2x[2x^2+1]]$

$= 2a e^{x^2+1}[4x+4x^3+2x]$

$= 2y[4x^3+6x] = 4yx[2x^2+3]$

3

[a] $\int \cot x \csc^3 x\,dx = \int -\csc^2 x\,.(-\csc x \cot x)\,dx$

$= \frac{-1}{3}\csc^3 x + c$

[b] $s = \sqrt{(x-1)^2+(y-0)^2}$

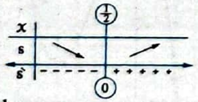
$, \because y = \sqrt{x}$ $\therefore y^2 = x$

$\therefore s = \sqrt{x^2-2x+1+y^2} = \sqrt{x^2-2x+1+x}$

$= \sqrt{x^2-x+1}$

$\therefore \frac{ds}{dx} = \frac{2x-1}{2\sqrt{x^2-x+1}}$

Putting : $\frac{ds}{dx} = 0$ $\therefore 2x-1 = 0$ $\therefore x = \frac{1}{2}$



$\therefore$ At $x = \frac{1}{2}$

$\therefore$ s has a minimum value.

$\therefore y = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$ $\therefore$ The point is : $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$

4

[a] $f(x) = \begin{cases} -x^2+4x, & -1 \le x \le 0 \\ x^2-4x, & 0 < x \le 3 \end{cases}$

$, \because f'(0^+) \ne f'(0^-)$

$\because f'(0)$ is not exist.

$\therefore f'(x) = \begin{cases} -2x+4, & -1 \le x < 0 \\ \text{is not exist}, & x = 0 \\ 2x-4, & 0 < x \le 3 \end{cases}$

$, f'(x) = 0$ in $-1 \le x < 0$

$\therefore -2x+4 = 0$ $, x = 2 \notin$ interval

$, f'(x) = 0$ in $0 < x \le 3$

$\therefore 2x-4 = 0$ $\therefore x = 2 \in$ interval

$\therefore f(-1) = -5$ $, f(2) = -4$ $, f(3) = -3$

$, f(0) = 0$

$\therefore f$ has absolute maximum value = zero at $x = 0$ and absolute minimum value = -5 at $x = -1$

[b] $\because \frac{dy}{dx} = 6x^2+bx$ $\therefore y = 2x^3+\frac{b}{2}x^2+c$

Putting : $x = 0$ $, y = 5$ $\therefore c = 5$

$\therefore y = 2x^3+\frac{b}{2}x^2+5$

Putting : $x = 2$ $, y = -3$

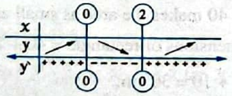
$\therefore -3 = 2(2)^3+\frac{b}{2}(2)^2+5$

$\therefore -3 = 21+2b$ $\therefore b = -12$

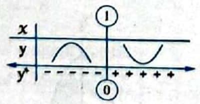
$\therefore f(x) = 2x^3-6x^2+5$

$\therefore f'(x) = 6x^2-12x$ $\therefore f''(x) = 12x-12$

put $f'(x) = 0$ $\therefore x = 0$ or 2

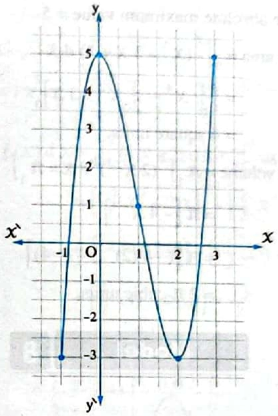


* Put $f''(x) = 0$ $\therefore x = 1$



* Determine other points : $f(-1) = -3$, $f(3) = 5$

x	-1	0	1	2	3
$f(x)$	-3	5	1	-3	5



5

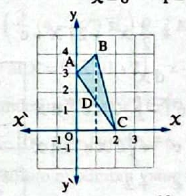
[a] let $y = \ln(9+x^3)$ $\therefore \frac{dy}{dx} = 3x^2 \times \frac{1}{9+x^3}$

let $z = x^2+3$ $\therefore \frac{dz}{dx} = 2x$

$\therefore \frac{dy}{dz} = \frac{dy}{dx} \div \frac{dz}{dx} = \frac{3x^2}{9+x^3} \times \frac{1}{2x} = \frac{3}{2} \times \frac{x}{9+x^3}$

$\therefore \left(\frac{dy}{dz}\right)_{x=1} = \frac{3}{2}\left(\frac{1}{10}\right) = \frac{3}{20}$

[b] The equation of $\overleftrightarrow{AB}$: $\frac{y-3}{x-0} = \frac{4-3}{1-0} = \frac{1}{1}$



$\therefore y-3 = x$ $\therefore y = x+3$

The equation of $\overleftrightarrow{BC}$: $\frac{y-0}{x-2} = \frac{4-0}{1-2} = -4$

$\therefore y = -4x+8$

The equation of $\overleftrightarrow{AC}$: $\frac{y-3}{x-0} = \frac{0-3}{2-0} = \frac{-3}{2}$

$\therefore y = \frac{-3}{2}x+3$

$\therefore$ Area of Δ ABC = area of Δ ABD + area of Δ DBC

$= \int_0^1 \left[(x+3)-\left(\frac{-3}{2}x+3\right)\right]dx + \int_1^2 \left[(-4x+8) - \left(\frac{-3}{2}x+3\right)\right]dx$

$= \int_0^1 \frac{5}{2}x\,dx + \int_1^2\left(\frac{-5}{2}x+5\right)dx = \left[\frac{5}{4}x^2\right]_0^1$

$+ \left[\frac{-5}{4}x^2+5x\right]_1^2 = \frac{5}{4}+\frac{5}{4} = \frac{5}{2}$ square unit.

If $y = \frac{-3}{2}x+3$ $\therefore x = \frac{-2}{3}y+2$

$\therefore$ The volume from rotation Δ AOC about y-axis

$= \pi\int_0^3 x^2\,dy = \pi\int_0^3\left(\frac{-2}{3}y+2\right)^2 dy$

$= \pi\left[\frac{-1}{2}\left(\frac{-2}{3}y+2\right)^3\right]_0^3 = 4\pi$ cubic units.

Model 7

1

(1) (b) (2) (a) (3) (a)
(4) (b) (5) (a) (6) (b)

2

[a] (1) $\int x \sin x\,dx$

$= -x\cos x - \int -\cos x\,dx$

$= -x\cos x + \sin x + c$

$\sin x$	x
$-\cos x$	1

(2) $\because f(x) = \sqrt{x^4+x^2}$ even function

$\therefore \int_{-1}^{1} f(x)\,dx = 2\int_0^1 \sqrt{x^4+x^2}\,dx$

$= 2\int_0^1 |x|\sqrt{x^2+1}\,dx$

$= \left[\frac{2}{3}(x^2+1)^{\frac{3}{2}}\right]_0^1$

$= \frac{2}{3}\left[2\sqrt{2}-1\right] = \frac{4\sqrt{2}-2}{3}$

[b] $y = \ln\left(2-\sqrt{2}\cos x\right)$, at $x = \frac{\pi}{4}$, then $y = 0$

$, \frac{dy}{dx} = \frac{\sqrt{2}\sin x}{2-\sqrt{2}\cos x}$

$\therefore$ Slope of tangent $= \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = 1$

$\therefore$ Equation of tangent is : $\frac{y-0}{x-\frac{\pi}{4}} = 1$

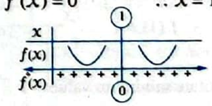
$\therefore y = x - \frac{\pi}{4}$

3

[a] $f(x) = (x-1)^4+3$ $\therefore f'(x) = 4(x-1)^3$

$, f''(x) = 12(x-1)^2$

Putting : $f'(x) = 0$ $\therefore x = 1$



The curve is convex downwards at $]-\infty , 1[$

$,]1 , \infty[$ and has no inflection points.

[b] Let the dimensions of cuboid are : (x , x , z)

$\because \frac{dx}{dt} = 0.4$ cm./sec. $, \frac{dz}{dt} = -0.5$ cm./sec.

$\therefore$ Volume $(v) = x^2 z$

$\therefore \frac{dv}{dt} = x^2 \frac{dz}{dt} + 2x \frac{dx}{dt} . z$

$= (6)^2 \times -0.5 + 2 (6) (0.4) (5) = 6$ cm³/sec.

4

[a] Put $z = x + 1$ $\quad \therefore x = z - 1$

$\therefore dx = dz$ at $x = 0$ $\quad \therefore z = 1$

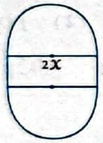
$, \text{at } x = 3$ $\quad \therefore z = 4$

$\int_1^4 (z-1) z^{\frac{1}{2}} \, dz = \int_1^4 \left(z^{\frac{3}{2}} - z^{\frac{1}{2}}\right) dz$

$= \left[\frac{2}{5} z^{\frac{5}{2}} - \frac{2}{3} z^{\frac{3}{2}}\right]_1^4$

$= \left[\frac{2}{5} (4)^{\frac{5}{2}} - \frac{2}{3} (4)^{\frac{3}{2}}\right]$

$- \left[\frac{2}{5} - \frac{2}{3}\right] = \frac{116}{15}$



[b] The perimeter = 400

$\therefore 2y + 2\pi x = 400$

$\therefore y = 200 - \pi x$

, let the area of playground (A) :

$\therefore A = 2xy + \pi x^2 = 2x(200 - \pi x) + \pi x^2$

$= 400x - 2\pi x^2 + \pi x^2 = 400x - \pi x^2$

$\therefore \grave{A} = 400 - 2\pi x$, putting : $\grave{A} = 0$

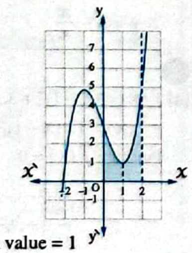
$\therefore x = \frac{200}{\pi}$, $\ddot{A} = -2\pi < 0$ maximum.

In the case of area is great as possible.

$\therefore x = \frac{200}{\pi}$ $\quad \therefore y = 200 - \pi \times \frac{200}{\pi} = 0$

$\therefore$ The figure becomes a circle with radius length $\frac{200}{\pi}$

5



[a] $f(x) = x^3 - 3x + 3$

$\therefore \grave{f}(x) = 3x^2 - 3$

, Putting : $\grave{f}(x) = 0$

$\therefore x = 1 , -1$

$, f(0) = 3 \quad , f(1) = 1$

$f(2) = 5$

$\therefore$ The absolute minimum value = 1

, the absolute maximum value = 5

The area $= \int_0^2 (x^3 - 3x + 3) \, dx$

$= \left[\frac{1}{4} x^4 - \frac{3}{2} x^2 + 3x\right]_0^2$

= 4 square units.

[b] The volume $= \pi \int_1^2 (2x^{-1})^2 \, dx = \pi \int_1^2 4x^{-2} \, dx$

$= \pi \left[-4x^{-1}\right]_1^2$

$= \pi \left[(-4(2)^{-1}) - (-4)\right]$

$= 2\pi$ cubic units.

Model 8

1

(1) $\frac{-3}{2}$ $\qquad$ (2) $7 \sec x \tan x \, e^{\sec x}$

(3) $(0 , -1)$ $\qquad$ (4) $\int_2^4 f(x) \, dx$

(5) $\frac{32}{3}$ $\qquad$ (6) $\frac{1}{2}$

2

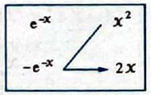
[a] (1) $\int \frac{(x+3)^3 - 27}{x} \, dx$

$= \int \frac{x^3 + 9x^2 + 27x + 27 - 27}{x} \, dx$

$= \int (x^2 + 9x + 27) \, dx$

$= \frac{1}{3} x^3 + \frac{9}{2} x^2 + 27x + c$

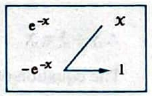
(2) $\int x^2 e^{-x} \, dx$



$= -x^2 e^{-x} - \int -2x e^{-x} \, dx$

$= -x^2 e^{-x} + \int 2x e^{-x} \, dx$

$= -x^2 e^{-x} + 2$



$[-x e^{-x} - \int -e^{-x} \, dx]$

$= -x^2 e^{-x} - 2x e^{-x}$

$-2e^{-x} + c = -e^{-x} [x^2 + 2x + 2] + c$

[b] $f\left(\frac{\pi}{4}\right) = 2 \tan^3 \left(\frac{\pi}{4}\right) = 2$ $\quad \therefore$ The point $\left(\frac{\pi}{4} , 2\right)$

$, \grave{f}(x) = 6 \tan^2 x . \sec^2 x$

$\therefore \grave{f}\left(\frac{\pi}{4}\right) = 6 \tan^2 \frac{\pi}{4} . \sec^2 \frac{\pi}{4} = 12$

$\therefore$ Equation of tangent is : $y - 2 = 12 \left(x - \frac{\pi}{4}\right)$

$\therefore y = 12x - 3\pi + 2$

3

[a] $\int_0^5 |x - 2| \, dx$

$f(x) = |x - 2| = \begin{cases} x - 2 & , x \geq 2 \\ -x + 2 & , x < 2 \end{cases}$

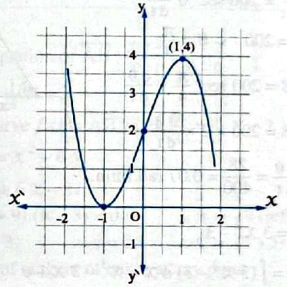
$\therefore \int_0^5 f(x) \, dx = \int_0^2 (-x + 2) \, dx + \int_2^5 (x - 2) \, dx$

$= \left[\frac{-1}{2} x^2 + 2x\right]_0^2 + \left[\frac{1}{2} x^2 - 2x\right]_2^5$

$= \left[(-2 + 4) - 0\right] + \left[\left(\frac{1}{2}(25) - 2(5)\right)\right]$

$- \left[\frac{1}{2}(4) - 2(2)\right] = \frac{13}{2}$

[b]



4

[a] $f(x) = 3\sqrt{4 - x^2}$

$\therefore \grave{f}(x) = \left(\frac{3}{2} (4 - x^2)^{\frac{-1}{2}}\right)(-2x) = \frac{-3x}{\sqrt{4 - x^2}}$

$, \grave{f}(x) = 0$ at $x = 0 \in [0 , 2]$

$, \grave{f}(x)$ is undefined at $x = 2 \in [0 , 2]$

$, x = -2 \notin [0 , 2] \quad , f(0) = 6 \quad , f(2) = 0$

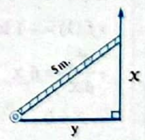
$\therefore f$ has absolute maximum value = 6 at $x = 0$

, has absolute miniumum value = 0 at $x = 2$

[b] $\frac{dx}{dt} = 1$ m./min.

, y the length of projection

of the rod on the floor.



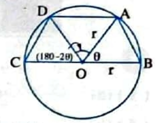
$\therefore x^2 + y^2 = 25$

$\therefore 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$

$\therefore \frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt}$, at $x = 3$ m. $\quad \therefore y = 4$

$\therefore \frac{dy}{dt} = \frac{-3}{4} \times 1 = \frac{-3}{4}$ m./min.

5



[a] Area of trapezoid

$= A(\Delta OBA) + A(\Delta OAD)$

$+ A(\Delta ODC)$

$= \frac{1}{2} r^2 \sin\theta + \frac{1}{2} r^2 \sin(180^\circ - 2\theta)$

$+ \frac{1}{2} r^2 \sin\theta = r^2 \sin\theta + \frac{1}{2} r^2 \sin 2\theta$

$\therefore \grave{A} = r^2 \cos\theta + r^2 \cos 2\theta$

$= r^2 \cos\theta + (2\cos^2\theta - 1) r^2$

, putting : $\grave{A} = 0$

$\therefore r^2 (2\cos^2\theta + \cos\theta - 1) = 0$

$\therefore r^2 (2\cos\theta - 1)(\cos\theta + 1)$

$\therefore \cos\theta = \frac{1}{2}$

or $\cos\theta = -1$ (refused)

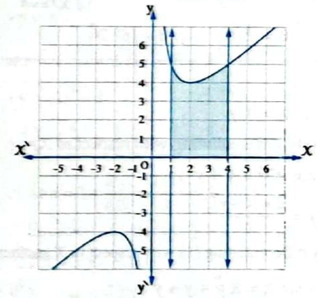
$\therefore \theta = 60^\circ$

$, \ddot{A}_{\theta = 60} > 0$ maximum

$\therefore$ The base angle $(\angle ABC)$

$= \frac{180^\circ - 60^\circ}{2} = 60^\circ$

[b]



$xy = 4 + x^2$ $\quad \therefore y = \frac{4}{x} + x$

(1) Area $= \int_1^4 \left(\frac{4}{x} + x\right) dx$

$= \left[4 \ln|x| + \frac{1}{2} x^2\right]_1^4$

$= 4 \ln 4 + \frac{15}{2}$ square unit.

(2) Volume $= \pi \int_1^4 \left(x + \frac{4}{x}\right)^2 dx$

$= \pi \int_1^4 \left(\frac{16}{x^2} + 8 + x^2\right) dx$

$= \pi \left[-16x^{-1} + 8x + \frac{1}{3} x^3\right]_1^4$

$= 57\pi$ cubic unit.

Model 9

1

(1) (a) (2) (d) (3) (a)

(4) (b) (5) (c) (6) (c)

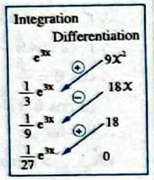
2

[a] (1) $\int \frac{3x}{x^2-1}\,dx = \frac{3}{2}\int \frac{2x}{x^2-1}\,dx$

$= \frac{3}{2}\ln|x^2-1| + c$

(2) $\int 9x^2 e^{3x}\,dx$

$= 3x^2e^{3x} - 2xe^{3x} + \frac{2}{3}e^{3x} + c$



[b] $y^3 = x^2$ $\therefore y = x^{\frac{2}{3}}$

$\therefore \frac{dy}{dx} = \frac{2}{3}x^{\frac{-1}{3}} = \frac{2}{3\sqrt[3]{x}}$

$\therefore$ Slope of tangent at $(x = 8) = \left(\frac{dy}{dx}\right)_{x=8} = \frac{1}{3}$

$\therefore \tan\theta = \frac{1}{3}$ $\therefore \theta = 18° \, 26'$

3

[a] $\because \sin x = xy$

(By differentiating with respect to x)

$\therefore \cos x = x\acute{y} + y$ (1)

$\therefore \acute{y} = \frac{\cos x - y}{x}$

(By differentiating (1) with respect to x another time)

$\therefore -\sin x = x\ddot{y} + \acute{y} + \acute{y}$

$\therefore -\sin x = x\ddot{y} + 2\acute{y}$

$\therefore -xy = x\ddot{y} + 2\times\left(\frac{\cos x - y}{x}\right)$

$\therefore -x^2y = x^2\ddot{y} + 2\cos x - 2y$

$\therefore x^2(y + \ddot{y}) + 2\cos x = 2y$

[b] $y = 2x^3 + 3x^2 + 4x + 5$

$\therefore \frac{dy}{dx} = 6x^2 + 6x + 4$

$\therefore$ Slope of tangent at $(-1, 2) = 4$

Putting : $\frac{dy}{dx} = 4$

$\therefore 6x^2 + 6x + 4 = 4$ $\therefore 6x(x+1) = 0$

$\therefore x = 0$ thus $y = 5$ or $x = -1$ thus $y = 2$

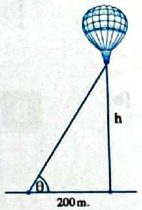
$\therefore$ The other point which its tangent has the same slope is $(0, 5)$

equation of other tangent : $\frac{y-5}{x-0} = 4$

$\therefore 4x - y + 5 = 0$

4

[a] $\tan\theta = \frac{h}{200}$



$\therefore h = 200\tan\theta$

$\therefore \frac{dh}{dt} = 200\sec^2\theta\,\frac{d\theta}{dt}$

at $h = 200$, $\theta = \frac{\pi}{4}$

$\therefore 28 = 200\sec^2\frac{\pi}{4}\times\frac{d\theta}{dt}$

$\therefore 28 = 200\times 2\times\frac{d\theta}{dt}$

$\therefore \frac{d\theta}{dt} = \frac{28}{400} = 0.07$ rad./min.

[b] $\frac{dy}{dx} = 3x^2 - 3$

$\therefore y = \int(3x^2-3)\,dx = x^3 - 3x + c$

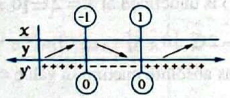
at $x = -2$ $\therefore y = -1$

$\therefore -1 = (-2)^3 - 3(-2) + c$

$\therefore c = 1$ $\therefore y = x^3 - 3x + 1$

Putting : $\frac{dy}{dx} = 0$ $\therefore 3(x^2-1) = 0$

$\therefore x = \pm 1$

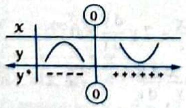


$\therefore f(-1) = 3$ local maximum value

, $f(1) = -1$ local minimum value

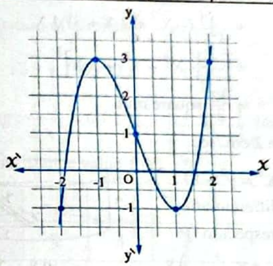
, $\frac{d^2y}{dx^2} = 6x$, put $\frac{d^2y}{dx^2} = 0$

$\therefore x = 0$



$\therefore (0, 1)$ inflection point , finding helping point $f(2) = 3$

x	-2	-1	0	1	2
y	-1	3	1	-1	3



5

(1) The equation of $\overleftrightarrow{AB}$: $\frac{y-2}{x-0} = \frac{4-2}{6-0} = \frac{1}{3}$

$\therefore y = \frac{1}{3}x + 2$

(2) The curve f cuts $\overleftrightarrow{AB}$ when $\frac{9}{x} = \frac{1}{3}x + 2$

$\therefore 27 = x^2 + 6x$

$\therefore x^2 + 6x - 27 = 0$

$\therefore (x+9)(x-3) = 0$ $\therefore x = -9$ (refused)

or $x = 3$ $\therefore y = 3$ $\therefore C = (3, 3)$

(3) Slope of tangent of curve at any point :

$\frac{dy}{dx} = \frac{-9}{x^2}$

$\therefore$ Slope of tangent at $C = -1$

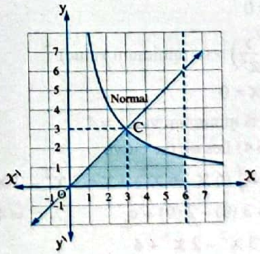
$\therefore$ Slope of normal $= 1$

$\therefore$ Equation of normal : $\frac{y-3}{x-3} = 1$

$\therefore x - y = 0$

$\therefore$ It passes through origin point.

(4)



The volume $= \pi\int_0^3 x^2\,dx + \pi\int_3^6 \left(\frac{9}{x}\right)^2 dx$

$= \pi\left[\frac{1}{3}x^3\right]_0^3 + \left[-81x^{-1}\right]_3^6$

$= 9\pi + \frac{27}{2}\pi = \frac{45}{2}\pi$ cubic unit.

Model 10

1

(1) e (2) $6\csc^2 x\,(5 - 2\cot x)^2$

(3) 3 (4) 44

(5) 0 (6) $\frac{16}{15}$

2

[a] (1) $\int \tan(3x+1)\,dx = \frac{-1}{3}\int \frac{-3\sin(3x+1)}{\cos(3x+1)}\,dx$

$= \frac{-1}{3}\ln|\cos(3x+1)| + c$

$= \frac{1}{3}\ln|\sec(3x+1)| + c$

(2) $\int(1-x^2)(3x-x^3)^5\,dx$

$= \frac{1}{3}\int(3-3x^2)(3x-x^3)^5\,dx$

$= \frac{1}{3}\times\frac{(3x-x^3)^6}{6} + c = \frac{1}{18}(3x-x^3)^6 + c$

[b] (1) $\because x = 2n^3 + 3$ $\therefore \frac{dx}{dn} = 6n^2$

, $y = n^4$ $\therefore \frac{dy}{dn} = 4n^3$

at $n = 1 : x = 2(1)^3 + 3 = 5$

, $y = (1)^4 = 1$ $\therefore$ The point $(5, 1)$

, $\frac{dy}{dx} = \frac{dy}{dn} \div \frac{dx}{dn} = \frac{4n^3}{6n^2} = \frac{2n}{3}$

$\therefore$ At $n = 1 : \frac{dy}{dx} = \frac{2(1)}{3} = \frac{2}{3}$

$\therefore$ Equation of tangent : $y - 1 = \frac{2}{3}(x - 5)$

$\therefore 2x - 3y - 7 = 0$

(2) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dn}\left(\frac{dy}{dx}\right)\times\frac{dn}{dx}$

$= \frac{d}{dn}\left(\frac{2n}{3}\right)\times\frac{1}{6n^2} = \frac{2}{3}\times\frac{1}{6n^2} = \frac{1}{9n^2}$

$\therefore$ At $n = 1$ $\therefore \frac{d^2y}{dx^2} = \frac{1}{9}$

3

[a] $f(x) = |x^3 - 1| = \begin{cases} -x^3 + 1 , & x < 1 \\ x^3 - 1 , & x \geq 1 \end{cases}$

$\acute{f}(1^-) = \lim_{h\to 0^-} \frac{-(1+h)^3 + 1 - 0}{h} = -3$

, $\acute{f}(1^+) = \lim_{h\to 0^+} \frac{(1+h)^3 - 1 - 0}{h} = 3$

$\therefore \grave{f}(1)$ is not exist

$$\therefore \grave{f}(x) = \begin{cases} -3x^2 & , \quad x < 1 \\ 3x^2 & , \quad x > 1 \end{cases}$$

$$, \grave{\grave{f}}(x) = \begin{cases} -6x & , \quad x < 1 \\ 6x & , \quad x > 1 \end{cases}$$

At $x < 1$ $\quad \grave{\grave{f}}(x) = 0 \quad \therefore -6x = 0$

$\therefore x = 0$, $\grave{\grave{f}}(x)$ changes before and after zero

, $\because$ It has only one tangent because $\grave{f}(0)$ exists

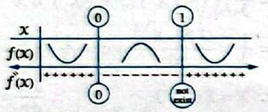
$\therefore$ At $x = 0$ it has inflection point $(0, 1)$

, at $x > 1$ $\quad \grave{\grave{f}}(x) = 0 \quad \therefore 6x = 0$

$\therefore x = 0$ doesn't satisfy

$\grave{\grave{f}}(x)$ is not exist at $x = 1$

$\therefore$ It has no tangent



The curve is convex upwards at $]0, 1[$

and convex downwards at $]-\infty, 0[$, $]1, \infty[$

The inflection points at $(0, 1)$

[b] $\int_{-2}^{5} [3f(x) - 6x]\, dx$

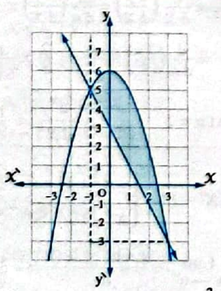
$= 3\int_{-2}^{5} f(x)\, dx - \int_{-2}^{5} 6x\, dx$

$= 3\left[\int_{2}^{3} f(x)\, dx + \int_{3}^{5} f(x)\, dx\right] - \int_{-2}^{5} 6x\, dx$

$= 3\,[9 + (-4)] - [3x^2]_{-2}^{5} = 15 - 63 = -48$

4

[a]



$y + x^2 = 6 \qquad \therefore y = -x^2 + 6 \qquad (1)$

$, y + 2x - 3 = 0 \qquad \therefore y = -2x + 3 \qquad (2)$

By solving (1) and (2) : $\therefore x = 3$ or $x = -1$

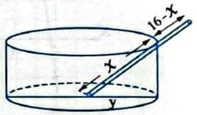
$\therefore$ Area $= \int_{-1}^{3} \left[(-x^2 + 6) - (-2x + 3)\right] dx$

$= \int_{-1}^{3} (-x^2 + 2x + 3)\, dx$

$= \left[-\frac{1}{3}x^3 + x^2 + 3x\right]_{-1}^{3}$

$= \frac{32}{3}$ square unit.

[b] $\frac{dx}{dt} = 2$ cm./sec.



, $\because x^2 = y^2 + 81$

(By differentiating with respect to t)

$\therefore 2x\frac{dx}{dt} = 2y\frac{dy}{dt} \qquad \therefore \frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$

When the rod reaches the base end.

$y = 12$ cm. , $x = 15$ cm.

$\therefore \frac{dy}{dt} = \frac{15}{12} \times 2 = \frac{5}{2}$ cm./sec.

5

[a] $\frac{d^2y}{dx^2} = 6 - 12x$

$\therefore \frac{dy}{dx} = \int (6 - 12x)\, dx = 6x - 6x^2 + c_1$

$\because$ There is a critical point at $x = 1$

$\therefore 0 = 6 - 6 + c_1 \qquad \therefore c_1 = 0$

$\therefore \frac{dy}{dx} = 6x - 6x^2$

Putting : $\frac{dy}{dx} = 0 \qquad \therefore x = 0$ or 1

at $x = 1$

$\therefore \left(\frac{d^2y}{dx^2}\right) < 0$ (maximum value)

at $x = 0$

$\therefore \left(\frac{d^2y}{dx^2}\right) > 0$ (minimum value)

$\therefore$ At $x = 0$

There is minimum value $= 4$

$\therefore (0, 4)$ lies on the curve

, $\because y = \int (6x - 6x^2)\, dx = 3x^2 - 2x^3 + c_2$

$\therefore 4 = 3(0) - 2(0) + c_2 \qquad \therefore c_2 = 4$

$\therefore y = 3x^2 - 2x^3 + 4$

$\because$ Slope of tangent at $x = -1$

is $\left(\frac{dy}{dx}\right)_{x=-1} = 6(-1) - 6(-1)^2 = -12$

$\therefore$ Slope of normal $= \frac{1}{12}$, at $x = -1$

$\therefore y = 3(-1)^2 - 2(-1)^3 + 4 = 9$

$\therefore$ The point is $(-1, 9)$

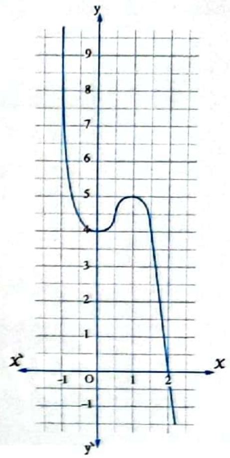
$\therefore$ Equation of normal $\frac{y - 9}{x + 1} = \frac{1}{12}$

$\therefore x - 12y + 109 = 0$

Putting : $\frac{d^2y}{dx^2} = 0 \qquad \therefore x = \frac{1}{2}$ (inflection)

Find assistance value $f(2) = 0$

x	-1	0	$\frac{1}{2}$	1	2
y	9	4	$\frac{9}{2}$	5	0



[b] $V = \pi\int_{0}^{1} y^2\, dx = \pi\int_{0}^{1} (x^3 + 1)^2\, dx$

$= \pi\int_{0}^{1} (x^6 + 2x^3 + 1)\, dx$

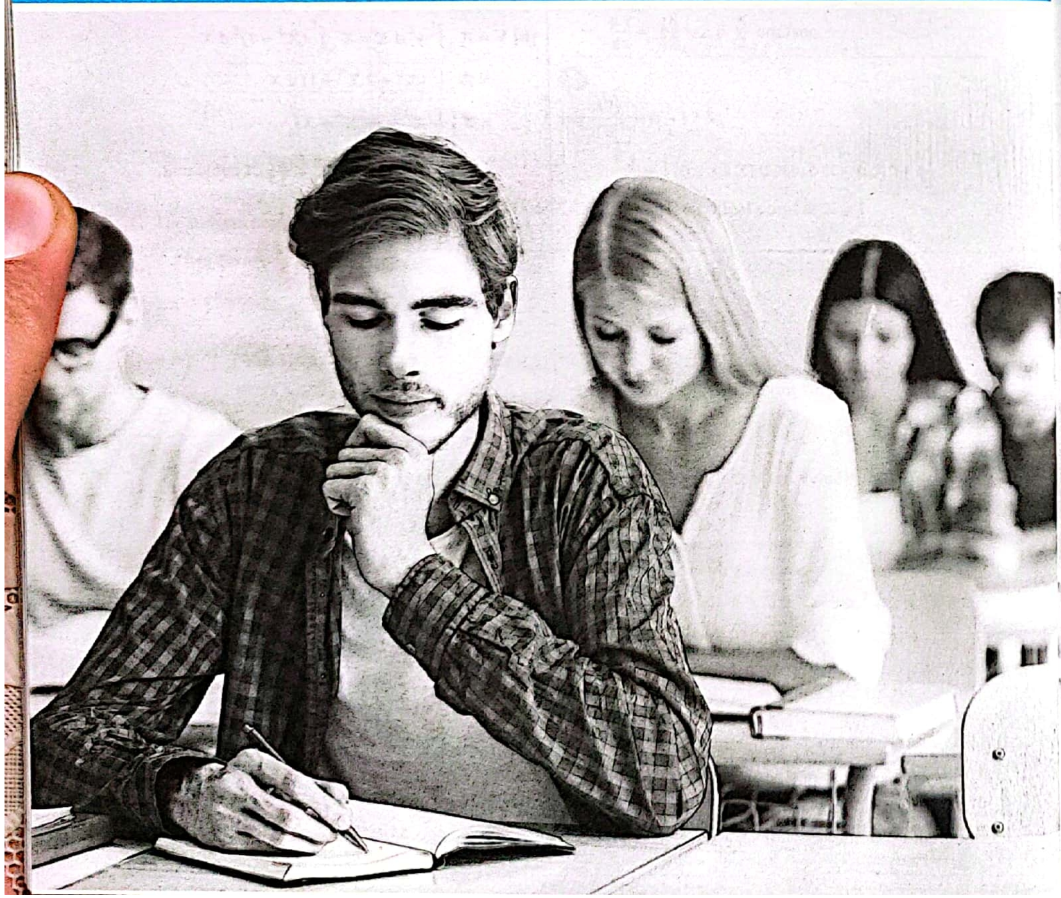
$= \pi\left[\frac{1}{7}x^7 + \frac{1}{2}x^4 + x\right]_{0}^{1}$

$= \pi\left[\frac{1}{7} + \frac{1}{2} + 1 - (0)\right] = \frac{23}{14}\pi$ cubic unit.

Answers

of

Egypt Exams



First session 2017

1 (a) **2** (b) **3** (d)

4

[a] $f(x) = (2 - x)\, e^{x}$

, $f'(x) = (2 - x)\, e^{x} + e^{x}(-1) = e^{x}(1 - x)$

, $f''(x) = e^{x}(-1) + (1 - x)\, e^{x} = -x\, e^{x}$

put $f'(x) = 0$ $\quad \therefore e^{x}(1 - x) = 0$

$\therefore 1 - x = 0$ $\quad \therefore x = 1$

, $\because f''(1) = -e$ (negative)

$\therefore$ There exist a local maximum value at $x = 1$

and its value $= f(1) = e$

[b] $\because f(x) = 3x^{4} - 4x^{3}$

$\therefore f'(x) = 12x^{3} - 12x^{2}$

put $f'(x) = 0$ $\quad \therefore 12x^{3} - 12x^{2} = 0$

$\therefore 12x^{2}(x - 1) = 0$

$\therefore x = 0 \in [-1, 2]$, $x = 1 \in [-1, 2]$

, $\because f(0) = \text{zero}$, $f(1) = -1$

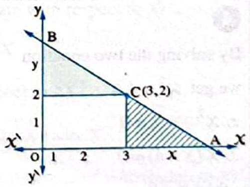
, $f(-1) = 7$, $f(2) = 16$

$\therefore$ The absolute maximum value $= 16$

and the absolute minimum value $= -1$

5 (a)

6



From similarity of the triangles

$\frac{x}{3} = \frac{2}{y}$ $\quad \therefore y = \frac{6}{x}$

The area of Δ AOB :

$A = \frac{1}{2}(x + 3)(y + 2)$

$= \frac{1}{2}(x + 3)\left(\frac{6}{x} + 2\right) = 6 + x + 9x^{-1}$

$\therefore A' = 1 - 9x^{-2}$ $\quad \therefore A'' = 18x^{-3}$

Putting $A' = 0$ $\quad \therefore 1 - \frac{9}{x^{2}} = 0$

$\therefore x^{2} = 9$, then $x = 3$

$A''_{x=3} > 0$ (i.e. minimum value)

$\therefore$ The smallest area : $A_{x=3} = 6 + 3 + \frac{9}{3}$

$= 12$ square unit.

7 (a)

8

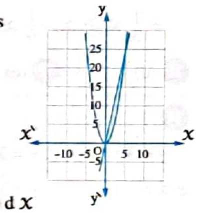
To find the intersection points

we put $x^{2} = 5x$

$\therefore x^{2} - 5x = 0$

$\therefore x(x - 5) = 0$

$\therefore x = 0$ or $x = 5$

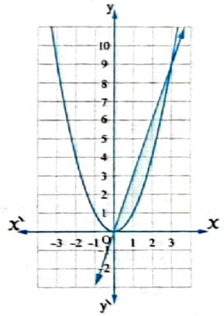


$\therefore$ The area $= \int_{0}^{5} (5x - x^{2})\, dx$

$= \left[\frac{5x^{2}}{2} - \frac{1}{3}x^{3}\right]_{0}^{5}$

$= \frac{125}{2} - \frac{125}{3} = \frac{125}{6}$ area unit.

9



To find the intersection points

we put $x^{2} = 3x$ $\quad \therefore x^{2} - 3x = 0$

$\therefore x(x - 3) = 0$

$\therefore x = 0$ or $x = 3$

$\therefore$ The volume $= \pi \int_{0}^{3} \left[(3x)^{2} - (x^{2})^{2}\right] dx$

$= \pi \int_{0}^{3} (9x^{2} - x^{4})\, dx$

$= \pi \left[3x^{3} - \frac{1}{5}x^{5}\right]_{0}^{3}$

$= \pi \left(81 - \frac{243}{5}\right) = \frac{162}{5}\pi$ volume unit.

10

[a] $\int \frac{x}{x+1}\,dx = \int \frac{x+1-1}{x+1}\,dx = \int \left(1 - \frac{1}{x+1}\right)dx$

$= x - \ln|x+1| + c$

[b] $\int x^2 \ln x \, dx$

x^2	$\ln x$
$\frac{1}{3}x^3$	$\frac{1}{x}$

$= \frac{1}{3}x^3 \ln x$

$- \frac{1}{3}\int x^2\,dx$

$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c$

11 (d) **12** (c) **13** (a)

14

$\because y = 3e^x \quad \therefore \acute{y} = 3e^x$

at $x = -1 \quad \therefore y = 3e^{-1} = \frac{3}{e}$

, $\acute{y} = 3e^{-1} = \frac{3}{e}$

$\therefore$ The slope of the tangent $= \frac{3}{e}$

and the slope of the normal $= \frac{-e}{3}$

$\therefore$ The equation of the normal is

$\left(y - \frac{3}{e}\right) = \frac{-e}{3}(x+1)$

i.e. $y = \frac{-e}{3}x - \frac{e}{3} + \frac{3}{e}$

15 (a) **16** (c)

17

$\because x \times y = \frac{z+1}{z-1} \times \frac{z-1}{z+1} = 1 \qquad \therefore y = \frac{1}{x} = x^{-1}$

$\therefore \acute{y} = -x^{-2} \qquad \therefore \ddot{y} = 2x^{-3}$

, at $z = 0 \qquad \therefore x = -1$

$\therefore \ddot{y} = 2(-1)^{-3} = -2$

18

$\because A = \pi r^2 \qquad \therefore \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

after 5 seconds $\qquad \therefore r = 4 \times 5 = 20$ cm.

$\therefore \frac{dA}{dt} = 2\pi \times 20 \times 4 = 160\pi$ cm²/sec.

Second session 2017

1 (b)

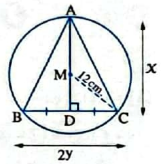
2

Let the triangle is ABC

, BC = 2 y cm.

, AD (hight of Δ) = x cm.

where each of x , $2y \in]0, 24[$



$\therefore A = \frac{1}{2}(2y) \cdot x = xy$

from properties of Δ MDC :

$y = \sqrt{12^2 - (x-12)^2} = \sqrt{24x - x^2}$

$\therefore A = x\sqrt{24x - x^2}$

$\acute{A} = \sqrt{24x - x^2} + x \times \frac{24 - 2x}{2\sqrt{24x - x^2}}$

$= \frac{24x - x^2 + 12x - x^2}{\sqrt{24x - x^2}} = \frac{36x - 2x^2}{\sqrt{24x - x^2}}$

let $\acute{A} = 0 \qquad \therefore x = 0$ (refused)

or $x = 18$

$\because$ The sign of $\acute{A}$ changed about two sides $x = 18$

$\therefore$ The area of a triangle is Max.

$[A]_{x=18} = 108\sqrt{3}$ cm².

3 (c)

4

By solving the two equation

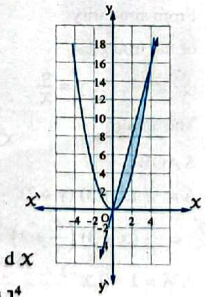
we get $x^2 = 4x$

$\therefore x^2 - 4x = 0$

$\therefore x(x-4) = 0$

, then $x = 0$

, $x = 4$



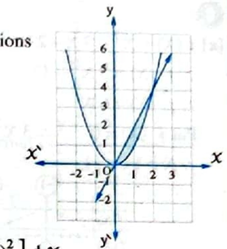
$\therefore$ The area $= \int_0^4 (4x - x^2)\,dx$

$= \left[2x^2 - \frac{1}{3}x^3\right]_0^4$

$= \frac{32}{3}$ square units.

5

By solving the two equations



$x^2 = 2x$

$\therefore x^2 - 2x = 0$

$\therefore x(x-2) = 0$

, then $x = 0$, $x = 2$

$\therefore$ The volume

$= \pi \int_0^2 \left[(2x)^2 - (x^2)^2\right]dx$

$= \pi \int_0^2 (4x^2 - x^4)\,dx$

$= \pi\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2 = \pi\left[\frac{32}{3} - \frac{32}{5}\right]$

$= \frac{64}{15}\pi$ cubic unit.

6

[a] $\int \frac{x}{3x^2+1}\,dx = \frac{1}{6}\int \frac{6x}{3x^2+1}\,dx$

$= \frac{1}{6}\ln|3x^2+1| + c$

[b] $\int \frac{x}{e^{2x}}\,dx = \int x \cdot e^{-2x}\,dx$

e^{-2x}	x
$\frac{-1}{2}e^{-2x}$	1

$= \frac{-1}{2}x e^{-2x} - \int \frac{-1}{2}e^{-2x}\,dx$

$= \frac{-1}{2}x e^{-2x} + \frac{1}{2}\left(\frac{-1}{2}e^{-2x}\right) + c$

$= \frac{-1}{2}x e^{-2x} - \frac{1}{4}e^{-2x} + c$

7 (d) **8** (a)

9

$y = x \sin x$ (Differentiate with respect to x)

$\therefore \frac{dy}{dx} = x\cos x + \sin x$

(Differentiate with respect to x)

$\therefore \frac{d^2y}{dx^2} = -x\sin x + \cos x + \cos x$

$= -y + 2\cos x$ (Differentiate with respect to x)

$\therefore \frac{d^3y}{dx^3} = -\frac{dy}{dx} - 2\sin x$ (by multiplying $\times x$)

$\therefore x\frac{d^3y}{dx^3} = -x\frac{dy}{dx} - 2x\sin x$

$\therefore x\frac{d^3y}{dx^3} + x\frac{dy}{dx} + 2y = 0$

10

Let the length and width of the rectangle

after time t sec. are

$L = 24 - 2t$, $w = 10 + 1.5t$

$\therefore$ The area (A) = (24 − 2t) (10 + 1.5 t)

$\therefore A = 240 + 16t - 3t^2$

$\therefore \frac{dA}{dt} = 16 - 6t$

After 4 sec. : $\frac{dA}{dt} = 16 - 6 \times 4 = -8$ cm²/sec.

the area stoped increasing when $\frac{dA}{dt} = 0$

$\therefore t = \frac{16}{6} = \frac{8}{3}$ sec.

11 (b) **12** (d) **13** (c)

14

$\because y = (x^3+5)^x$ (by taking logarithm of both sides)

$\therefore \ln y = x\ln(x^3+5)$

$\frac{1}{y}\frac{dy}{dx} = \ln(x^3+5) + x \times \frac{3x^2}{x^3+5}$

$\therefore \frac{dy}{dx} = (x^3+5)^x\left[\ln(x^3+5) + \frac{3x^3}{x^3+5}\right]$

15 (b) **16** (b) **17** (a)

18

[a] $f(x) = x^3 - 3x^2 - 9x$

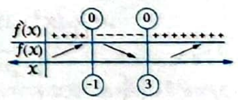
$\therefore \acute{f}(x) = 3x^2 - 6x - 9$

let $\acute{f}(x) =$ zero

$\therefore 3x^2 - 6x - 9 = 0$

$\therefore 3(x-3)(x+1) = 0$

$\therefore x = 3$, $x = -1$



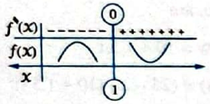
$\therefore$ Local minimum value $f(3) = -27$

$\therefore$ Local maximum value $f(-1) = 5$

$\because f''(x) = 6x - 6$

let $f''(x) = 0$

$\therefore x = 1$



$\therefore$ The inflection point $(1 , -11)$

[b] $f(x) = 10 x e^{-x}$

$\therefore f'(x) = -10 x e^{-x} + 10 e^{-x} = 10 e^{-x}(1 - x)$

let $f'(x) = 0$

$\therefore 10 e^{-x}(1 - x) = 0$

$\therefore 1 - x = 0 \qquad \therefore x = 1$

x	zero	1	4
$f(x)$	zero	$\frac{10}{e}$	$\frac{40}{e4}$

- Absolute maximum value $= \frac{10}{e} = 3.68$
- Absolute minimum value = zero.

First session 2018

1 (b) 2 (c)

3

[a] $\int x^3 (x^2+1)^6 \, dx$

Put $z = x^2 + 1 \qquad \therefore dz = 2x\,dx$

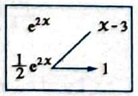
$\therefore x\,dx = \frac{1}{2} dz$

$$\therefore \int x^3 (x^2+1)^6 dx = \int x^2 . x (x^2+1)^6 dx$$
$$= \int (z-1) z^6 \times \frac{1}{2} dz$$
$$= \int \left(\frac{1}{2} z^7 - \frac{1}{2} z^6\right) dz$$
$$= \frac{1}{16} z^8 - \frac{1}{14} z^7 + c$$
$$= \frac{1}{16}(x^2+1)^8 - \frac{1}{14}(x^2+1)^7 + c$$

[b] $\int (x-3) e^{2x} dx$

$= \frac{1}{2}(x-3) \times e^{2x} - \int \frac{1}{2} e^{2x} dx$

$= \frac{1}{2}(x-3) e^{2x} - \frac{1}{4} e^{2x} + c$



4 (a) 5 (b)

6

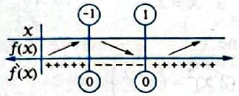
[a] $f(x) = x^3 - 3x - 2$

$\therefore f'(x) = 3x^2 - 3$

$\therefore f''(x) = 6x$

Put $f'(x) = 0 \qquad \therefore 3x^2 - 3 = 0$

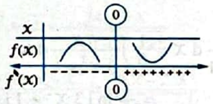
$\therefore x = 1$ or -1



$\therefore$ The function has local maximum value at $x = -1$, $f(-1) = $ zero

, the function has local minimum value at $x = 1$, $f(1) = -4$

Put $f''(x) = 0 \qquad \therefore 6x = 0 , x = 0$



, $f(0) = -2$

there exist an inflection point at $(0 , -2)$

[b] $f(x) = x(x^2 - 12) = x^3 - 12x$

$\therefore f'(x) = 3x^2 - 12$

let $f'(x) = 0 \quad \therefore x = 2 \in [-1 , 4]$

or $x = -2 \notin [-1 , 4]$

$\because f(-1) = 11 , f(2) = -16 , f(4) = 16$

then the function has absolute maximum value = 16 at $x = 4$, and has absolute minimum value = -16 at $x = 2$

7 (a) 8 (b)

9

$x = \sec\theta \qquad \therefore \frac{dx}{d\theta} = \sec\theta\tan\theta$

, $y = \tan\theta \qquad \therefore \frac{dy}{d\theta} = \sec^2\theta$

$\therefore \frac{dy}{dx} = \frac{\sec^2\theta}{\sec\theta\tan\theta} = \csc\theta$

$\therefore \left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{6}} = 2 \qquad \therefore$ Slope of the tangent = 2

, slope of the normal $= \frac{-1}{2}$

at $\theta = \frac{\pi}{6}$ then : $x = \frac{2\sqrt{3}}{3}$, $y = \frac{\sqrt{3}}{3}$

$\therefore$ The point $\left(\frac{2\sqrt{3}}{3} , \frac{\sqrt{3}}{3}\right)$

Equation of the tangent : $\frac{y - \frac{\sqrt{3}}{3}}{x - \frac{2\sqrt{3}}{3}} = 2$

i.e. $2x - y - \sqrt{3} = 0$

$\therefore$ Equation of the normal : $\frac{y - \frac{\sqrt{3}}{3}}{x - \frac{2\sqrt{3}}{3}} = \frac{-1}{2}$

i.e. $3x + 6y - 4\sqrt{3} = 0$

10

$\sin y + \cos 2x = 0$ (by differentiate with respect to x)

$\therefore \cos y \frac{dy}{dx} - 2\sin 2x = 0$ (by differentiate with respect to x)

$\therefore \cos y \frac{d^2y}{dx^2} + \frac{dy}{dx} \times - \sin y \times \frac{dy}{dx} - 4\cos 2x = 0$ (divide by cos y)

$\therefore \frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 \tan y = 4\cos 2x \sec y$

11 (b) 12 (b)

13

$y = a x^b$ (by differentiate with respect to t)

$\therefore \frac{dy}{dt} = a b x^{b-1} \frac{dx}{dt}$

Multiply by $\left(\frac{1}{y}\right)$

$\therefore \frac{1}{y} \times \frac{dy}{dt} = \frac{1}{y} \times \frac{a b x^b}{x} \times \frac{dx}{dt}$

$\therefore \frac{1}{y} \times \frac{dy}{dt} = \frac{1}{y} \times \frac{b}{x} \times y \times \frac{dx}{dt}$

$\therefore \frac{1}{y} \times \frac{dy}{dt} = \frac{b}{x} \times \frac{dx}{dt}$

Another solution : (by taking natural logarithm of both sides)

$\therefore \ln y = \ln a + b \ln x$

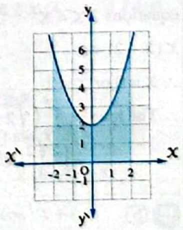
(by differentiate with respect to t)

$\therefore \frac{1}{y} \times \frac{dy}{dt} = \frac{b}{x} \times \frac{dx}{dt}$

14

The volume

$$= \pi \int_{-2}^{2} y^2 dx$$
$$= \pi \int_{-2}^{2} (x^2+2)^2 dx$$
$$= \pi \int_{-2}^{2} (x^4 + 4x^2 + 4) dx$$
$$= \pi \left[\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x\right]_{-2}^{2}$$
$$= \frac{752}{15} \pi \text{ cubic unit.}$$



15 (b) 16 (d)

17

Let the radius length of the circle = r

, length of arc = l

$\therefore$ area of sector $= \frac{1}{2} l r = 4$

$\therefore l = \frac{8}{r}$

let the perimeter of the sector (P)

$P = 2r + l = 2r + \frac{8}{r} = 2r + 8r^{-1}$

$\therefore P' = 2 - 8r^{-2}$, $P'' = 16 r^{-3}$

Put $P' = 0$

$\therefore r = \pm 2$ (negative value is refused)

$\therefore P''(2) = 2$

$\therefore$ P has minimum value at $r = 2$

, $l = \frac{8}{r} = \frac{8}{2} = 4$, $\theta^{rad} = \frac{l}{r} = \frac{4}{2} = 2^{rad}$

18

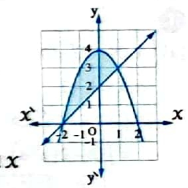
$y = 4 - x^2$, $y = x + 2$

by solving the two equations

$\therefore x = 1$ or $x = -2$

$\therefore$ The area

$$= \int_{-2}^{1} \left[(4 - x^2) - (x + 2)\right] dx$$
$$= \int_{-2}^{1} (2 - x^2 - x) dx$$
$$= \left[2x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right]_{-2}^{1}$$
$$= \frac{9}{2} \text{ square unit.}$$



Second session 2018

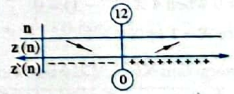
1 ⓐ 2 ⓓ

3

(1) $z(n) = 20\left[\frac{n}{12} - \ln\left(\frac{n}{12}\right)\right] + 30$

$\therefore z'(n) = 20\left[\frac{1}{12} - \frac{12}{n} \times \frac{1}{12}\right] = 20\left(\frac{n-12}{12n}\right)$

$\therefore z'(n) = $ zero when $n = 12$ days



Minimum value after 12 days

(2) The least number of bacteria

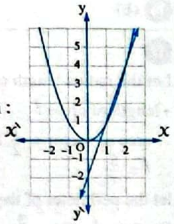
$= 20\left[\frac{12}{12} - \ln\left(\frac{12}{12}\right)\right] + 30 = 50$ in each cm^3.

4

$y_1 = x^2$

, $y_2 = 3x - 2$

, to get the point of intersection :



$x^2 = 3x - 2$

$\therefore x^2 - 3x + 2 = 0$

$\therefore x = 1$ or $x = 2$

$\therefore$ The volume $= \pi \int_1^2 (y_2^2 - y_1^2)\, dx$

$= \pi \int_1^2 \left[(3x-2)^2 - (x^2)^2\right] dx$

$= \pi \left[\frac{(3x-2)^3}{3 \times 3} - \frac{1}{5}x^5\right]_1^2$

$= \frac{4}{5}\pi$ cubic unit.

5 ⓒ **6** ⓑ

7

[a] $\int x(x+2)^6\, dx$ putting $z = x + 2$

$\therefore x = z - 2$ $\therefore dx = dz$

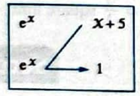
$\therefore \int x(x+2)^6\, dx$

$= \int (z-2) z^6\, dz = \int z^7 - 2z^6\, dz$

$= \frac{1}{8}z^8 - \frac{2}{7}z^7 + c$

$= \frac{1}{8}(x+2)^8 - \frac{2}{7}(x+2)^7 + c$

[b] $\int (x+5)e^x\, dx$



$= (x+5)e^x - \int e^x\, dx$

$= (x+5)e^x - e^x + c$

$= xe^x + 4e^x + c$

8 ⓓ **9** ⓑ

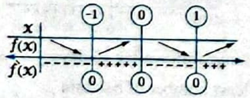
10

[a] $f(x) = x^4 - 2x^2$

$\therefore f'(x) = 4x^3 - 4x$

, $f'(x) = 0$ when $4x(x^2 - 1) = 0$

$\therefore x = 0$ or $x = 1$ or $x = -1$



$\therefore$ The function has a local minimum value

at $x = -1$, $f(-1) = -1$

, the function has a local maximum value at $x = 0$

, $f(0) = 0$

, the function has a local minimum value

at $x = 1$, $f(1) = -1$

[b] $f(x) = \frac{4x}{x^2 + 1}$

$\therefore f'(x) = \frac{(x^2+1) \times (4) - 4x \times 2x}{(x^2+1)^2} = \frac{-4x^2 + 4}{(x^2+1)^2}$

, $f'(x) = 0$ at $x = 1 \in [-1 , 3]$

or at $x = -1 \in [-1 , 3]$

$\therefore f(-1) = -2$, $f(1) = 2$, $f(3) = \frac{6}{5}$

$\therefore$ The function has absolute maximum value $= 2$

and absolute minimum value $= -2$

11 ⓓ **12** ⓑ

13

$y = 3 + \sec x$ $\therefore \frac{dy}{dx} = \sec x \tan x$

$\therefore \left(\frac{dy}{dx}\right)_{(x = \frac{2\pi}{3})} = 2\sqrt{3}$ $\therefore$ Slope of tangent $= 2\sqrt{3}$

, slope of normal $= \frac{-1}{2\sqrt{3}}$

, when $x = \frac{2\pi}{3}$ $\therefore y = 1$

$\therefore$ The point is $\left(\frac{2\pi}{3} , 1\right)$

$\therefore$ The equation of tangent : $\frac{y - 1}{x - \frac{2\pi}{3}} = 2\sqrt{3}$

i.e. $2\sqrt{3}x - y - \frac{4}{3}\sqrt{3}\pi + 1 = 0$

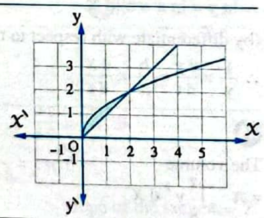
, equation of normal : $\frac{y - 1}{x - \frac{2\pi}{3}} = \frac{-1}{2\sqrt{3}}$

i.e. $x + 2\sqrt{3}y - 2\sqrt{3} - \frac{2}{3}\pi = 0$

14

$y_1 = \sqrt{2x}$, $y_2 = x$

By solving the two equations $2x = x^2$



$x(x - 2) = 0$

$\therefore x = 0$ or $x = 2$

$\therefore$ The area $= \int_0^2 \left(\sqrt{2x} - x\right) dx$

$= \left[\frac{2\sqrt{2}}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2\right]_0^2 = \frac{2}{3}$ square unit.

15 ⓑ **16** ⓓ

17

$\because \sin x = xy$ (Differentiate with respect to x)

$\therefore \cos x = xy' + y$

Differentiate with respect to x another time

$-\sin x = xy'' + y' + y'$ $\therefore -\sin x = xy'' + 2y'$

$\therefore -xy = xy'' + 2 \times \left(\frac{\cos x - y}{x}\right)$

$\therefore -x^2 y = x^2 y'' + 2\cos x - 2y$

$\therefore x^2(y + y'') + 2\cos x = 2y$

18

$xe^y = 2 - \ln 2 + \ln x$

Differentiate with respect to "t"

$\therefore xe^y \frac{dy}{dt} + e^y \frac{dx}{dt} = \frac{1}{x} \times \frac{dx}{dt}$

$\therefore \frac{dx}{dt} = 6$ at $x = 2$, $y = 0$

$\therefore 2 \times e^0 \times \frac{dy}{dt} + e^0 \times 6 = \frac{1}{2} \times 6$

$\therefore 2 \times \frac{dy}{dt} = -3$

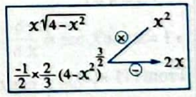
$\therefore \frac{dy}{dt} = -\frac{3}{2}$

First session 2019

1 (c) **2** (b)

3

[a] $\int x^3\sqrt{4 - x^2}\, dx$



$= \int x^2 \times x\sqrt{4 - x^2}\, dx$

$= -\frac{1}{3}x^2(4 - x^2)^{\frac{3}{2}}$

$-\int -\frac{2}{3}x(4 - x^2)^{\frac{3}{2}}\, dx$

$= -\frac{1}{3}x^2(4 - x^2)^{\frac{3}{2}} - \frac{1}{3}\int\left[-2x(4 - x^2)^{\frac{3}{2}}\right] dx$

$= -\frac{1}{3}x^2(4 - x^2)^{\frac{3}{2}} - \frac{2}{15}(4 - x^2)^{\frac{5}{2}} + c$

[b] $\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx$

$= \int \left[(1 - \cos^2 x)\sin x\right] dx$

$= \int \left[\sin x - \cos^2 x \sin x\right] dx$

$= -\cos x + \frac{1}{3}\cos^3 x + c$

4 (d) **5** (a)

6

[a] $y = x^3 + ax^2 + bx$

$\therefore y' = 3x^2 + 2ax + b$

$\therefore y'' = 6x + 2a$

, $\because$ (3 , −9) is an inflection point.

$\therefore 0 = 6(3) + 2a$ $\therefore a = -9$

, $\because$ the point (3 , −9) $\in$ the curve.

$\therefore -9 = 27 + 9a + 3b$, by substitution the value of a

$\therefore b = 15$

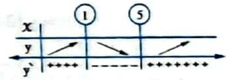
$\therefore$ The equation of the curve :

$y = x^3 - 9x^2 + 15x$

$\therefore y' = 3x^2 - 18x + 15$

, putting $y' = 0$

$\therefore x = 1$ or 5



$\therefore$ The function has a local maximum value at

$x = 1$, $f(1) = 7$

, the function has a local minimum value at $x = 5$

, $f(5) = -25$

[b] $f(x) = 2x^2 e^x$

$\therefore f'(x) = 2x^2 e^x + 4xe^x = 2xe^x(x + 2)$

, $f'(x) = 0$ $\therefore x = 0 \in [-3 , 1]$

, $x = -2 \in [-3 , 1]$

$\because f(0) = 0$, $f(-2) = \frac{8}{e^2} = 1.08$

, $f(-3) = \frac{18}{e^3} \approx 0.896$, $f(1) = 2e \approx 5.4$

$\therefore$ The function has absolute maximum value

$= 2e$ at $(x = 1)$

and absolute minimum value $= 0$ at $(x = 0)$

7 (c) **8** (a)

9

$y = 3x^2 - \ln x$

$\therefore \frac{dy}{dx} = 6x - \frac{1}{x}$

$\therefore \left(\frac{dy}{dx}\right)_{x=1} = 6 - 1 = 5$

$\therefore$ The equation of tangent : $\frac{y-3}{x-1} = 5$

i.e. $5x - y - 2 = 0$

10

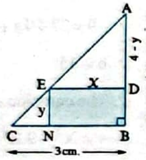
let $BD = y$

, $BN = x$

$\therefore AD = 4 - y$

, $CN = 3 - x$

, $\because \overline{ED} \,//\, \overline{BC}$



$\therefore \frac{x}{3} = \frac{4-y}{4}$

$\therefore x = \frac{12-3y}{4} = 3 - \frac{3}{4}y$ (1)

$\therefore$ The area of the rectangle (A) $= y\,x$

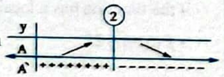
$= y\left(3 - \frac{3}{4}y\right)$

$= 3y - \frac{3}{4}y^2$

$\therefore \frac{dA}{dy} = 3 - \frac{3}{2}y$, putting $\frac{dA}{dy} = 0$

$\therefore y = 2$

$\therefore$ The area of the rectangle is maximum at $y = 2$



, from equation (1)

$\therefore x = 3 - \frac{3}{4} \times 2 = \frac{3}{2}$

$\therefore$ The dimensions of the rectangle are : 2 cm. , $\frac{3}{2}$ cm.

11 (b) **12** (d)

13

Volume of the sphere (v) $= \frac{4}{3}\pi r^3$

$\therefore \frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\therefore$ The evaporation rate is proportional to the surface area

$\because \frac{dv}{dt} \propto A$

$\therefore \frac{dv}{dt} = KA$ (where K is constant)

$\therefore 4\pi r^2 \frac{dr}{dt} = k \times 4\pi r^2$

$\therefore \frac{dr}{dt} = K$

i.e. The radius of the raindrop decreases at a constant rate.

14

$y\,x = 10 - \cos x$ (Differentiate with respect to x)

$\therefore y + x\frac{dy}{dx} = \sin x$

(Differentiate with respect to x another time)

$\therefore \frac{dy}{dx} + x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \cos x$

$\therefore x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = \cos x$

15 (a) **16** (d)

17

$\because \frac{dy}{dx} = a\csc^2 x$ (by integrating with respect to x)

$\therefore y = -a\cot x + c$

, $\because$ the curve passes through the points $\left(\frac{\pi}{4}, 5\right)$ and $\left(\frac{3\pi}{4}, 1\right)$

$\therefore 5 = -a + c$ (1)

, $1 = a + c$ (2)

From (1) , (2) :

$\therefore c = 3$, $a = -2$

$\therefore$ The equation $y = 2\cot x + 3$

18

$\because |x-4| = \begin{cases} x-4 & , x \ge 4 \\ -x+4 & , x < 4 \end{cases}$

, $\because f$ is continuous at $x = 4$

$\therefore \int_0^6 |x-4|\,dx = \int_0^4 (-x+4)\,dx + \int_4^6 (x-4)\,dx$

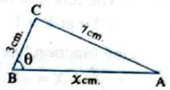
$= \left[-\frac{1}{2}x^2 + 4x\right]_0^4 + \left[\frac{1}{2}x^2 - 4x\right]_4^6$

$= [8 - 0] + [-6 - (-8)] = 10$

Second session 2019

1 (a) **2** (a)

3



$\because (AC)^2 = (3)^2 + x^2 - 2 \times 3 \times x \times \cos\theta$

$\therefore 49 = 9 + x^2 - 6x\cos\theta$ (1)

(by differentiating with respect to t)

$\therefore 0 = 2x\frac{dx}{dt} - 6x\left(-\sin\theta\frac{d\theta}{dt}\right) - 6\frac{dx}{dt}\cos\theta$

$\therefore 2x\frac{dx}{dt} + 6x\sin\theta\frac{d\theta}{dt} - 6\cos\theta\frac{dx}{dt} = 0$ (2)

at $\theta = \frac{\pi}{3}$ from equation (1) :

$\therefore 49 = 9 + x^2 - 6x\cos 60°$

$\therefore x^2 - 3x - 40 = 0$

$\therefore (x-8)(x+5) = 0$

$\therefore x = 8$ or $x = -5$ (refused)

, substituting in equation (2) :

$\therefore 2 \times 8 \times \frac{dx}{dt} + 6 \times 8 \times \sin 60° \times 1.3 - 6\cos 60°\frac{dx}{dt} = 0$

$\therefore 13\frac{dx}{dt} = -31.2\sqrt{3}$

$\therefore \frac{dx}{dt} = -2.4\sqrt{3}$ cm./min.

4

$y = \sec x$ $\therefore \frac{dy}{dx} = \sec x\tan x$

$\therefore \frac{d^2y}{dx^2} = \sec x\sec^2 x + \sec x\tan x\tan x$

$= \sec^3 x + \sec x\tan^2 x$

$= \sec x(\sec^2 x + \tan^2 x)$

$\therefore y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \sec x \,.\, \sec x(\sec^2 x + \tan^2 x) + \sec^2 x\tan^2 x$

$= \sec^2 x(\sec^2 x + 2\tan^2 x)$

$= \sec^2 x(3\sec^2 x - 2)$

$= y^2(3y^2 - 2)$

5 (b) **6** (d)

7

$y = \ln\left(2 - \sqrt{2}\cos x\right)$, when $x = \frac{\pi}{4}$, then $y = 0$

, $\frac{dy}{dx} = \frac{\sqrt{2}\sin x}{2 - \sqrt{2}\cos x}$

$\therefore$ The slope of tangent $= \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = 1$

$\therefore$ The equation of tangent is : $\frac{y-0}{x-\frac{\pi}{4}} = 1$

, then $x - y - \frac{\pi}{4} = 0$

8

Let the number be x

$\therefore y = \frac{1}{x} + 4x^2$

$\therefore \grave{y} = -x^{-2} + 8x = \frac{-1+8x^3}{x^2}$

, $\grave{\grave{y}} = 2x^{-3} + 8$

, $\grave{y} = 0$ $\therefore 8x^3 = 1$ $\therefore x = \frac{1}{2}$

, $(\grave{\grave{y}})_{x=\frac{1}{2}} = 24$ (positive)

$\therefore$ at $x = \frac{1}{2}$ the sum is the smallest possible.

9 (d) **10** (a)

11

The slope of the tangent $= \frac{dy}{dx} = x\sqrt{x+1}$

$\therefore y = \int x\sqrt{x+1}\,dx$

$= \int (x+1-1)(x+1)^{\frac{1}{2}}\,dx$

$= \int \left[(x+1)^{\frac{3}{2}} - (x+1)^{\frac{1}{2}}\right]dx$

$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c$

$\because \left(0, \frac{11}{15}\right)$ lies on the curve.

$\therefore \frac{11}{15} = \frac{2}{5}(0+1)^{\frac{5}{2}} - \frac{2}{3}(0+1)^{\frac{3}{2}} + c$ $\therefore c = 1$

$\therefore$ The equation of the curve :

$y = \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + 1$

12

$f(0) = f(0^+) = f(0^-)$

i.e. The function is continuous at $x = 0$

$$\therefore \int_{-1}^{3} f(x) = \int_{-1}^{0} (2x + x^2)\, dx + \int_{0}^{3} (2x - x^2)\, dx$$

$$= \left[x^2 + \frac{1}{3}x^3\right]_{-1}^{0} + \left[x^2 - \frac{1}{3}x^3\right]_{0}^{3}$$

$$= \left[(0) - \left((-1)^2 + \frac{1}{3}(-1)^3\right)\right]$$

$$+ \left[(3)^2 - \frac{1}{3}(3)^3 - (0)\right] = -\frac{2}{3}$$

13 (c) **14** (d)

15

[a] $\because y = a x^3 + b x^2$

$\therefore y' = 3 a x^2 + 2 b x$

, $y'' = 6 a x + 2 b$

$\because$ (1 , 4) is an inflection point

$\therefore y'' = 0$ at $x = 1$

$\therefore 6a + 2b = 0 \qquad \therefore 3a + b = 0 \qquad (1)$

$\because$ The curve passes through the point (1 , 4)

$\therefore y = 4$ at $x = 1$

$\therefore a + b = 4 \qquad (2)$

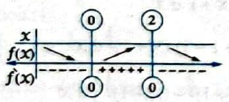
From 1 , 2 : $a = -2$, $b = 6$

$\therefore y = -2x^3 + 6x^2$

$\therefore y' = -6x^2 + 12x$

, putting $y' = 0 \qquad \therefore -6x(x - 2) = 0$

$\therefore x = 0$ or $x = 2$



$\therefore$ The function has a local minimum value at $x = 0$, $f(0) = 0$

, the function has a local maximum value at $x = 2$, $f(2) = 8$

[b] $f(x) = 2x e^x$

$f'(x) = 2x e^x + 2e^x = 2e^x (x + 1)$

, $f'(x) = 0 \qquad \therefore 2e^x (x + 1) = 0$

$\therefore x + 1 = 0$

, $x = -1 \in [-3 , 1]$

, $\because f(-3) = \frac{-6}{e^3} \approx -0.3$, $f(-1) = \frac{-2}{e} \approx -0.74$

, $f(1) = 2e \approx 5.4$

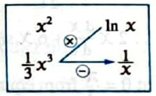
$\therefore$ The function has absolute maximum value $= 2e$ at $x = 1$

, the function has absolute minimum value $= \frac{-2}{e}$ at $x = -1$

16 (b) **17** (c)

18

[a] $\int x^2 \ln x\, dx$



$= \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \times \frac{1}{x}\, dx$

$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\, dx$

$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c$

[b] $\int x \sin(2x^2)\, dx = \frac{1}{4}\int 4x \sin(2x^2)\, dx$

$= -\frac{1}{4}\cos(2x^2) + c$

First session 2020

1 (b) **2** (c)

3

$\frac{dy}{dx} = x e^y = \frac{x}{e^{-y}}$

$\therefore \int x\, dx = \int e^{-y}\, dy \qquad \therefore \frac{1}{2}x^2 + c = -e^{-y}$

$\because$ The curve passes through the point (1 , 0)

$\therefore \frac{1}{2} + c = -1 \qquad \therefore c = -\frac{3}{2}$

$\therefore$ The equation of the curve is : $\frac{1}{2}x^2 - \frac{3}{2} = -e^{-y}$

$\therefore x^2 + 2e^{-y} - 3 = 0$

4

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (\sin x + \cos x)^2\, dx$$

$$= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x + 2\sin x \cos x)\, dx$$

$$= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (1 + \sin 2x)\, dx = \left[x - \frac{1}{2}\cos 2x\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$

$$= \left[\frac{\pi}{2} + \frac{1}{2}\right] - \left[-\frac{\pi}{2} + \frac{1}{2}\right] = \pi$$

5 (a) **6** (d)

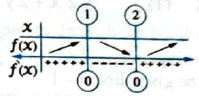
7

[a] $f(x) = 2x^3 - 9x^2 + 12x$

$f'(x) = 6x^2 - 18x + 12$

, $f'(x) = 0 \qquad \therefore 6(x - 1)(x - 2) = 0$

$\therefore x = 1$ or $x = 2$



$\therefore f$ is increasing on $]2 , \infty[$, $]-\infty , 1[$

, decreasing on $]1 , 2[$

, the function has a local maximum value at $x = 1$, $f(1) = 5$

, the function has a local minimum value at $x = 2$, $f(2) = 4$

[b] $f(x) = x^4 - 6x^2 + 16$

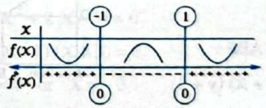
$\therefore f'(x) = 4x^3 - 12x$

$f''(x) = 12x^2 - 12$

Putting $f''(x) = 0$

$\therefore 12(x^2 - 1) = 0$

$\therefore x = \pm 1$



The curve is convex downwards at $]-\infty , -1[$, $]1 , \infty[$, convex upwards at $]-1 , 1[$

and the inflection points (1 , 11) , (– 1 , 11)

8 (b) **9** (d)

10

[a] $\int (x^2 + 1)\sqrt{x + 2}\, dx$

Putting $z = \sqrt{x + 2}$

$\therefore x = z^2 - 2$, $dx = 2z\, dz$

$\therefore \int (x^2 + 1)\sqrt{x + 2}\, dx$

$= \int \left((z^2 - 2)^2 + 1\right) \times z \times 2z\, dz$

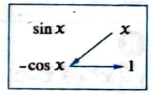
$= \int 2z^2 (z^4 - 4z^2 + 5)\, dz$

$= \int (2z^6 - 8z^4 + 10z^2)\, dz$

$= \frac{2}{7}z^7 - \frac{8}{5}z^5 + \frac{10}{3}z^3 + c$

$= \frac{2}{7}(x + 2)^{\frac{7}{2}} - \frac{8}{5}(x + 2)^{\frac{5}{2}} + \frac{10}{3}(x + 2)^{\frac{3}{2}} + c$

[b] $\int x \sin x\, dx$

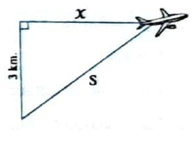


$= -x \cos x + \int \cos x\, dx$

$= -x \cos x + \sin x + c$

11 (b) **12** (d)

13



$\frac{dx}{dt} = 480$ km/h.

$x^2 + (3)^2 = s^2$

$\therefore 2x \frac{dx}{dt} = 2s \frac{ds}{dt} \quad (1)$

at t = 30 sec. $= \frac{30}{60 \times 60} = \frac{1}{120}$ hr.

$\therefore x = \frac{1}{120} \times 480 = 4$ km. , s = 5 km.

from equation (1) :

$4 \times 480 = 5 \times \frac{ds}{dt} \qquad \therefore \frac{ds}{dt} = 384$ km/h.

14

$y = a e^{x^2 + 1}$

$\frac{dy}{dx} = 2x a e^{x^2 + 1} = 2xy$

$\frac{d^2y}{dx^2} = 2x \frac{dy}{dx} + 2y$

$= (2x)(2xy) + 2y = 4x^2 y + 2y$

$= 2y(2x^2 + 1)$

$\frac{d^3y}{dx^3} = 2\frac{dy}{dx}(2x^2 + 1) + 2y \times 4x$

$= 4xy(2x^2 + 1) + 8xy$

$= 8x^3 y + 12xy = 4xy(2x^2 + 3)$

Another solution :

$\frac{dy}{dx} = 2x a e^{x^2 + 1}$

$\frac{d^2y}{dx^2} = 2ae^{x^2+1} + 4x^2 \times ae^{x^2+1}$

$= 2ae^{x^2+1}(1+2x^2)$

$\frac{d^3y}{dx^3} = 4xae^{x^2+1}(1+2x^2) + 2ae^{x^2+1}(4x)$

$= 4xae^{x^2+1}(1+2x^2+2)$

$= 4xy(3+2x^2)$

---

15 (a) **16** (c)

17

$y = x^3 + 3x - 2$

The slope of tangent $= \frac{dy}{dx} = 3x^2 + 3$

, the slope of the given line $= \frac{-1}{6}$

$\therefore$ The slope of the required line $= 6$

$\therefore 3x^2 + 3 = 6$ $\therefore x^2 = 1$

$\therefore x = \pm 1$

$\therefore$ The points of tangency are $(1, 2)$, $(-1, -6)$

$\therefore$ The equation of tangents are :

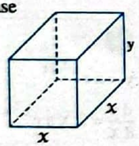
$\frac{y-2}{x-1} = 6$ *i.e.* $y - 6x + 4 = 0$

, $\frac{y+6}{x+1} = 6$ *i.e.* $y - 6x = 0$

18

Let the side length of the squared base $= x$ cm.

, the height $= y$ cm.



$\therefore$ The total area $= 2x^2 + 4xy$

$\therefore 2x^2 + 4xy = 150$

$\therefore y = \frac{150 - 2x^2}{4x}$

$\therefore$ The volume of the cuboid (V)

$= x^2 y = \frac{x^2(150 - 2x^2)}{4x} = \frac{75}{2}x - \frac{1}{2}x^3$

$\therefore \frac{dV}{dx} = \frac{75}{2} - \frac{3}{2}x^2$

$, \because \frac{dV}{dx} = 0$ $\therefore \frac{3}{2}x^2 = \frac{75}{2}$

$\therefore x = 5$

, when $x = 5$, then $\frac{d^2V}{dx^2} < 0$

$\therefore$ The volume is maximum

$\therefore$ The maximum volume (V)

$= \frac{75}{2} \times 5 - \frac{1}{2} \times 5^3 = 125 \text{ cm}^3$

Second session 2020

1 (b) **2** (d)

3

$\because x^2 + y^2 = 8$ (1) $\therefore 2x + 2y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{-x}{y}$

, the slope of the given line $= -1$

, the slope of the required line $= 1$

$\therefore \frac{-x}{y} = 1$ $\therefore y = -x$

From (1) : $\therefore x^2 + (-x)^2 = 8$

$\therefore x^2 = 4$ $\therefore x = \pm 2$

$\therefore$ The points of tangency are :

$(2, -2)$, $(-2, 2)$

The equations of tangents are :

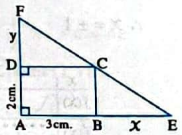
$\frac{y+2}{x-2} = 1$ *i.e.* $y - x + 4 = 0$

or $\frac{y-2}{x+2} = 1$ *i.e.* $y - x - 4 = 0$

4

$\Delta BEC \sim \Delta DCF$

$\therefore \frac{x}{3} = \frac{2}{y}$, $x, y > 0$



$\therefore y = \frac{6}{x}$

Area of ΔAEF :

$A = \frac{1}{2}(3+x)(y+2)$

$= \frac{1}{2}(3+x)\left(2 + \frac{6}{x}\right) = 6 + x + \frac{9}{x}$

$\therefore \frac{dA}{dx} = 1 - 9x^{-2}$

Putting $\frac{dA}{dx} = 0$

$\therefore x^2 = 9$ $\therefore x = 3$

$, \frac{d^2A}{dx^2} = 18x^{-3}$

at $x = 3$ $\left(\frac{d^2A}{dx^2}\right)_{x=3} > 0$ (minimum value)

$\therefore$ The smallest area of the triangle $= 6 + 3 + \frac{9}{3}$

$= 12 \text{ cm}^2$

5 (c) **6** (c)

7

$\frac{dy}{dx} = \frac{1+x}{xy}$

$\therefore \int y\,dy = \left(\frac{1+x}{x}\right)dx$

$\therefore \int y\,dy = \int\left(\frac{1}{x} + 1\right)dx$

$\therefore \frac{1}{2}y^2 = \ln|x| + x + c$

$\because$ The curve passes through the point $(1, 2)$

$\therefore 2 = 0 + 1 + c$ $\therefore c = 1$

$\therefore$ The equation of the curve is :

$\frac{1}{2}y^2 = \ln|x| + x + 1$

i.e. $y^2 = 2\ln|x| + 2x + 2$

8

$\int_{-\pi}^{\pi}(4 + \pi\cos 2x)\,dx = \left[4x + \frac{\pi}{2}\sin 2x\right]_{-\pi}^{\pi}$

$= (4\pi) - (-4\pi) = 8\pi$

---

9 (c) **10** (a)

11

[a] $f(x) = x^3 + 3x^2 - 9x - 7$

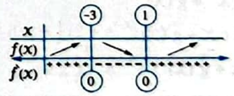
$, f'(x) = 3x^2 + 6x - 9$

Putting $f'(x) = 0$

$\therefore 3(x^2 + 2x - 3) = 0$

$\therefore 3(x-1)(x+3) = 0$

$\therefore x = 1$ or $x = -3$



$\therefore f$ is increasing on $]-\infty, -3[$, $]1, \infty[$

, decreasing on $]-3, 1[$

, the function has a local maximum value at

$x = -3$, $f(-3) = 20$

, the function has a local minimum value at

$x = 1$, $f(1) = -12$

[b] $f(x) = x^3 - x^2$

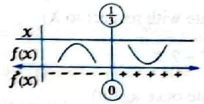
$f'(x) = 3x^2 - 2x$

$f''(x) = 6x - 2$

, putting $f''(x) = 0$

$\therefore 6x - 2 = 0$

$\therefore x = \frac{1}{3}$



$\therefore$ The curve is convex downwards at $]\frac{1}{3}, \infty[$

, convex upwards at $]-\infty, \frac{1}{3}[$

and the inflection point at $\left(\frac{1}{3}, \frac{-2}{27}\right)$

---

12 (b) **13** (c)

14

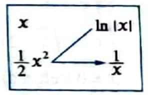
[a] $\int x(x+2)^6\,dx$

$\int(x+2-2)(x+2)^6\,dx$

$= \int(x+2)^7\,dx - \int 2(x+2)^6\,dx$

$= \frac{1}{8}(x+2)^8 - \frac{2}{7}(x+2)^7 + c$

[b] $\int x\ln|x|\,dx$



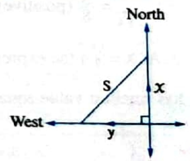
$= \frac{1}{2}x^2\ln|x| - \int\frac{1}{2}x\,dx$

$= \frac{1}{2}x^2\ln|x| - \frac{1}{4}x^2 + c$

---

15 (b) **16** (a)

17



$\frac{dx}{dt} = 30$ km./h.

$, \frac{dy}{dt} = 80$ km./h.

$, s^2 = x^2 + y^2$

$\therefore 2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$ (1)

after one hour from the movement of the second car :

$y = 1 \times 80 = 80$ km.

$, x = 2 \times 30 = 60$ km. , then $s = 100$ km.

From (1) :

$100 \times \frac{ds}{dt} = 60 \times 30 + 80 \times 80$

$\therefore \frac{ds}{dt} = 82$ km./h.

18

$y = e^{3x} + x^2$

(by differentiate with respect to x)

$\therefore \frac{dy}{dx} = 3e^{3x} + 2x$

(by differentiate once again)

$\therefore \frac{d^2y}{dx^2} = 9e^{3x} + 2$

but $(e^{3x} = y - x^2)$

$\therefore \frac{d^2y}{dx^2} = 9(y - x^2) + 2$

First session 2021

1 (c)

Solution :

Let $y = \sqrt{x} + \frac{1}{x}$

$\therefore \acute{y} = \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$, putting $\acute{y} = 0$

$\therefore \frac{1}{2\sqrt{x}} = \frac{1}{x^2}$ $\qquad \therefore x^2 = 2\sqrt{x}$

$\therefore \sqrt{x}\left(x\sqrt{x} - 2\right) = 0$

$\therefore x = 0$ (refused) or $x^{\frac{3}{2}} = 2$

$\therefore x = (2)^{\frac{2}{3}} = \sqrt[3]{4}$

, $\because \ddot{y} = -\frac{1}{4}x^{-\frac{3}{2}} + 2x^{-3}$

$(\ddot{y})_{x=\sqrt[3]{4}} = \frac{3}{8}$ (positive)

$\therefore$ At $x = \sqrt[3]{4}$ the expression $\left(\sqrt{x} + \frac{1}{x}\right)$

has smallest value equals $\frac{3}{\sqrt[3]{4}}$

2 (a)

Solution :

$\int \frac{x}{x^2 - 2x + 1}\,dx = \int \frac{x}{(x-1)^2}\,dx$

$= \int x(x-1)^{-2}\,dx = \int (x - 1 + 1)(x-1)^{-2}\,dx$

$= \int (x-1)^{-1}\,dx + \int (x-1)^{-2}\,dx$

$= \int \frac{1}{x-1}\,dx + \int (x-1)^{-2}\,dx$

$= \ln|x - 1| - (x-1)^{-1} + c$

$= \ln|x - 1| - \frac{1}{x-1} + c$

3 (d)

Solution :

$\because f(x) = \sin x + \frac{1}{2}\sin 2x$

$\therefore \acute{f}(x) = \cos x + \cos 2x$

, putting $\acute{f}(x) = 0$

$\therefore \cos 2x + \cos x = 0$

x		$\frac{\pi}{3}$	
$f(x)$	↗		↘
$\acute{f}(x)$	+ + + + + + + +	0	− − − − − − − −

$\therefore 2\cos^2 x + \cos x - 1 = 0$

$\therefore (2\cos x - 1)(\cos x + 1) = 0$

$\therefore \cos x = \frac{1}{2}$, then $x = \frac{\pi}{3}$

or $\cos x = -1$, then $x = \pi$ (refused)

$\therefore$ At $x = \frac{\pi}{3}$ the function has local maximum value.

4 (c)

Solution :

$\because f(x) = \sqrt{x^2 + 9}$

$\therefore \acute{f}(x) = \frac{2x}{2\sqrt{x^2+9}}$

$= \frac{x}{\sqrt{x^2+9}}$

, putting $\acute{f}(x) = 0$ $\qquad \therefore x = 0$

, $\because f(0) = 3$, $f(-3) = f(3) = 3\sqrt{2}$

$\therefore$ The absolute minimum value $= 3$

5 (c)

Solution :

$\because f(x) = cx^2 + g(x)$

$\therefore \acute{f}(x) = 2cx + \acute{g}(x)$

, $\ddot{f}(x) = 2c + \ddot{g}(x)$

, $\because$ (1 , 5) is an inflection point to the curve of the function f

$\therefore \ddot{f}(1) = 0$ $\qquad \therefore 2c + \ddot{g}(1) = 0$

$\therefore 2c = -6$ $\qquad \therefore c = -3$

, $f(1) = 5$ $\qquad \therefore 5 = c + g(1)$

$\therefore c + k = 5$ $\qquad \therefore -3 + k = 5$

$\therefore k = 8$ $\qquad \therefore k - c = 8 - (-3) = 11$

6 (b)

Solution :

$\because y = \ln(\tan x)$ $\qquad \therefore \acute{y} = \frac{\sec^2 x}{\tan x}$

, $\because$ the x-coordinate of the tangency point $= \frac{\pi}{4}$

, $\because f\left(\frac{\pi}{3}\right) = \ln\tan\left(\frac{\pi}{4}\right) = 0$

$\therefore$ The tangency point $= \left(\frac{\pi}{4}, 0\right)$

$\therefore [\acute{y}]_{x=\frac{\pi}{4}} = 2$

$\therefore$ The slope of normal $= \frac{-1}{2}$

$\therefore$ The equation of normal $= \frac{y - 0}{x - \frac{\pi}{4}} = \frac{-1}{2}$

i.e. $8y + 4x = \pi$

7 (c)

Solution :

$\because$ The function is even , $\int_{-5}^{5} f(x)\,dx = 2a$

$\therefore \int_{0}^{5} f(x)\,dx = \int_{-5}^{0} f(x)\,dx = a$

, $\int_{0}^{3} f(x)\,dx = \int_{-3}^{0} f(x)\,dx = b$

$\therefore \int_{-5}^{-3} f(x)\,dx = \int_{-5}^{0} f(x)\,dx - \int_{-3}^{0} f(x)\,dx$

$= a - b$

8 (a)

Solution :

$\because f(x) = x^3 - 3x - 4$ is decreasing

, $\acute{f}(x) = 3x^2 - 3$ $\qquad \therefore \acute{f}(x) < 0$

$\therefore 3x^2 - 3 < 0$ $\qquad \therefore 3x^2 < 3$

$\therefore x^2 < 1$ $\qquad \therefore |x| < 1$

9 (b)

Solution :

$\because \int \frac{2e^x + 1}{e^x}\,dx = \int (2 + e^{-x})\,dx$

$= 2x - e^{-x} + C$

10 (a)

Solution :

$\because \frac{dx}{d\theta} = -2\sin 2\theta$, $\frac{dy}{d\theta} = 3\cos 3\theta$

$\therefore \frac{dy}{dx} = \frac{3\cos 3\theta}{-2\sin 2\theta}$

$\therefore \left(\frac{dy}{dx}\right)_{\theta=0} = \frac{3}{0}$ (undefined)

$\therefore$ The tangent to the curve is parallel to y-axis.

11 (a)

Solution :

$\because P = f(t)$, $\frac{dp}{dt} \propto \frac{1}{t}$ $\qquad \therefore \frac{dp}{dt} = \frac{a}{t}$

$\therefore \int dp = \int \frac{a}{t}\,dt$ $\qquad \therefore p = a\ln|t| + c$

, $\because f(1) = 200$ $\qquad \therefore 200 = c$

$\therefore p = a\ln|t| + 200$

, $\because f\left(\sqrt[3]{e}\right) = \frac{700}{3}$ $\qquad \therefore \frac{700}{3} = a\ln e^{\frac{1}{3}} + 200$

$\therefore \frac{700}{3} = \frac{1}{3}a + 200$ $\qquad \therefore a = 100$

$\therefore p = 100\ln|t| + 200$

, at $t = 3$

$\therefore f(3) = 100\ln 3 + 200 = 100(\ln 3 + 2)$

$= 100(\ln 3 + \ln e^2)$

$= 100\ln(3e^2)$

12 (c)

Solution :

$y = \sqrt{2(\sec x + \tan x)}$

$\frac{dy}{dx} = \frac{2(\sec x\tan x + \sec^2 x)}{2\sqrt{2(\sec x + \tan x)}} = \frac{\sec x(\tan x + \sec x)}{\sqrt{2(\sec x + \tan x)}}$

$\therefore \frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{\sec x(\tan x + \sec x)}{2(\sec x + \tan x)} = \frac{1}{2}\sec x$

13 (c)

Solution :

$\int \frac{2}{x\ln x^2}\,dx = \int \frac{2}{2x\ln|x|}\,dx = \int \frac{\frac{1}{x}}{\ln|x|}\,dx$

$= \ln\left|\ln|x|\right| + c$

14 (a)

Solution :

Let the length of side of the hexagon $= \ell$ cm.

$\therefore$ The side length of the shaded triangle $= \frac{1}{\sqrt{3}}\ell$

$\therefore$ The area of the shaded region (A)

$= 6 \times \frac{1}{2} \times \left(\frac{1}{\sqrt{3}}\ell\right)^2 \sin 60^\circ = \frac{\sqrt{3}}{2}\ell^2$

$\therefore \frac{dA}{dt} = \sqrt{3}\,\ell\,\frac{d\ell}{dt}$

, at $\ell = 4$ cm. , $\frac{d\ell}{dt} = \sqrt{3}$ cm./sec.

$\therefore \frac{dA}{dt} = \sqrt{3} \times 4 \times \sqrt{3} = 12$ cm²/sec.

15 (c)

Solution :

Let $z = \frac{1}{y} = (x+1)^2$, $k = x^2$

$\therefore \frac{dz}{dk} = \frac{dz}{dx} \div \frac{dk}{dx} = \frac{2(x+1)}{2x} = 1 + \frac{1}{x}$

at $x = 1$ $\qquad \therefore \frac{dz}{dk} = 2$

16 (d)

Solution :

$\because f''(x) > 0$ for each $x \neq 3$

$\therefore$ The curve is convex downwards for each $x \neq 3$

, $\because f''(3)$ undefined

$\therefore$ The figure (d) can represent the curve of the function f

17 (a)

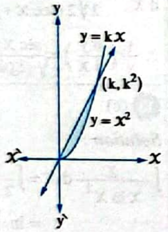
Solution :

$\because y = e^{x^3-4}$ $\qquad \therefore y' = 3x^2 e^{x^3-4}$

$\therefore 2y' = 6x^2 y$ $\qquad \therefore m = 6$

18 (b)

Solution :



Putting $x^2 = kx$

$\therefore x^2 - kx = 0$

$\therefore x(x-k) = 0$

$\therefore x = 0$ or $x = k$

$\therefore$ The point of intersections are $(0 , 0)$, (k , k^2)

$\therefore \int_0^k (kx - x^2)\,dx = \frac{9}{2}$ $\qquad \therefore \left[\frac{1}{2}kx^2 - \frac{1}{3}x^3\right]_0^k = \frac{9}{2}$

$\therefore \frac{1}{2}k^3 - \frac{1}{3}k^3 = \frac{9}{2}$ $\qquad \therefore \frac{1}{6}k^3 = \frac{9}{2}$

$\therefore k^3 = 27$ $\qquad \therefore k = 3$

19 (b)

Solution :

$\int_0^1 \left[a\sqrt{x} - (ax - 2)\right] dx = \frac{13}{6}$

$\therefore \left[\frac{2}{3}a x^{\frac{3}{2}} - \frac{1}{2}a x^2 + 2x\right]_0^1 = \frac{13}{6}$

$\therefore \frac{2}{3}a - \frac{1}{2}a + 2 = \frac{13}{6}$

$\therefore \frac{1}{6}a = \frac{1}{6}$ $\qquad \therefore a = 1$

20 (c)

Solution :

$\because xy = x^y$ $\qquad \therefore \ln(xy) = y \ln x$

$\therefore \ln x + \ln y = y \ln x$ $\qquad \therefore \frac{1}{x} + \frac{y'}{y} = \frac{y}{x} + y' \ln x$

at $(1 , 1)$ $\qquad \therefore 1 + y' = 1 + 0$

$\therefore y' = 0$

$\therefore$ The equation of tangent is $\frac{y-1}{x-1} = 0$

$\therefore y - 1 = 0$

21 (b)

Solution :

$\because x\frac{dy}{dx} - y = 3$ $\qquad \therefore x\frac{dy}{dx} = y + 3$

$\therefore \frac{dy}{dx} = \frac{y+3}{x}$ $\qquad \therefore \int \frac{dy}{y+3} = \int \frac{dx}{x}$

$\therefore \ln|y+3| = \ln|x| + c$

at $x = 1$, $y = -2$ $\qquad \therefore c = 0$

$\therefore \ln|y+3| = \ln|x|$

i.e. $|y+3| = |x|$

22 (c)

Solution :

$\because xy = 3$ $\qquad \therefore x = \frac{3}{y}$

$\pi \int_1^k \left(\frac{3}{y}\right)^2 dy = 6\pi$ $\qquad \therefore \int_1^k 9y^{-2}\,dy = 6$

$\left[-9y^{-1}\right]_1^k = 6$ $\qquad \therefore \frac{-9}{k} + 9 = 6$

$\therefore \frac{-9}{k} = -3$ $\qquad \therefore k = 3$

23 (d)

Solution :

$\lim_{x\to 0} \frac{e^{x+a} - e^a}{x} = \lim_{x\to 0} e^a\left(\frac{e^x - 1}{x}\right)$

$= e^a \times \lim_{x\to 0} \frac{e^x - 1}{x} = e^a$

$\therefore e^a = \frac{1}{e} = e^{-1}$ $\qquad \therefore a = -1$

24 (a)

Solution :

$\int_0^4 y\,dx = 68$ $\qquad \therefore \int_0^4 (x^3 + a)\,dx = 68$

$\therefore \left[\frac{1}{4}x^4 + ax\right]_0^4 = 68$ $\qquad \therefore 64 + 4a = 68$

$\therefore 4a = 4$ $\qquad \therefore a = 1$

25 (a)

Solution :

$\because f'(x) = 0$ at $x = 2$ and its sign changes from positive to negative

$\therefore$ The function f has a local maximum value at $x = 2$

$\therefore$ (b) is true

, $\because$ the slope of the drawn curve is negative for each $x < 4$

$\therefore f''(x)$ is negative for each $x < 4$

, $\because$ the slope of the drawn curve is positive for each $x > 4$

$\therefore f''(x)$ is positive for each $x > 4$

$\therefore$ The curve of the function is convex upwards in $]-\infty , 4[$

, convex downward in $]4 , \infty[$

$\therefore$ (c) is true

, $\because f'(x)$ is positive for each $x < 2$

$\therefore$ The function f is increasing for each $x < 2$

, $\because -2 > -3$ $\qquad \therefore f(-2) > f(-3)$

$\therefore$ (d) is true

, $\because$ at $x = 4$ the function has no tangent

$\therefore$ There's no inflection point in spite of the changing of the convexity of the curve at $x = 4$

$\therefore$ (a) is wrong.

Second session 2021

1 (d)

Solution :

$y = \cot 5x$

$\because \frac{dy}{dx} = -5\csc^2 5x$

$\therefore \frac{dy}{dx} + 5y^2 = -5\csc^2 5x + 5\cot^2 5x$

$= -5(\csc^2 5x - \cot^2 5x = -5 \times 1 = -5$

2 (c)

Solution :

$\lim_{x\to 0} \frac{\ln(1+x)^{4a}}{x} = \lim_{x\to 0} \frac{4a\ln(1+x)}{x} = 4a \times 1$

$\therefore a^2 + 4 = 4a$ $\qquad \therefore a^2 - 4a + 4 = 0$

$\therefore (a-2)^2 = 0$ $\qquad \therefore a = 2$

3 (a)

Solution :

$\int 4\sin 2x \,.\, e^{\cos 2x}\,dx$

$= -2\int -2\sin 2x \,.\, e^{\cos 2x}\,dx$

$= -2e^{\cos 2x} + c$

4 (a)

Solution :

$\int \frac{2}{y}\,dy = \int \frac{1}{x}\,dx$

$\therefore 2\ln|y| = \ln|x| + c$ $\qquad \therefore \ln y^2 = \ln|x| + c$

5 (c)

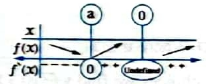
Solution :

$f(x) = x^2 + \frac{2a^3}{x}$

$\therefore f'(x) = 2x - \frac{2a^3}{x^2} = \frac{2x^3 - 2a^3}{x^2}$

Putting $f'(x) = 0$

$\therefore 2x^3 - 2a^3 = 0$ $\qquad \therefore x = a$



$\therefore f$ is decreasing on $]-\infty , a[$

6 (b)

Solution :

$\because z = x + 2$ $\qquad \therefore x = z - 2$

$dx = dz$

$\therefore \int \frac{x-2}{x^2+4x+4}\,dx = \int \frac{z-4}{z^2}\,dz$

$= \int \left(\frac{1}{z} - 4z^{-2}\right) dz$

$= \ln|z| + 4z^{-1} + c = \ln|z| + \frac{4}{z} + c$

7 (a)

Solution :

$3y^2\frac{dy}{dx} = 2x + 1$

$\therefore \int 3y^2\,dy = \int (2x+1)\,dx$

$\therefore y^3 = x^2 + x + c$

$\because x = 1$ at $y = -1$

$\therefore (-1)^3 = 1 + 1 + c$ $\qquad \therefore c = -3$

$\therefore$ The relation is : $y^3 = x^2 + x - 3$

8 (b)

Solution :

$x = \tan(1 + y)$ "differentiate with respect to x"

$\therefore 1 = \sec^2(1 + y)\dfrac{dy}{dx}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\sec^2(1+y)} = \dfrac{1}{1 + \tan^2(1+y)} = \dfrac{1}{1 + x^2}$

at $x = 2$ $\qquad \therefore \dfrac{dy}{dx} = 0.2$

9 (c)

Solution :

$2x + 1 = \sin y$ "differentiate with respect to x"

$\therefore 2 = \cos y \dfrac{dy}{dx}$ $\qquad \therefore \dfrac{dy}{dx} = \dfrac{2}{\cos y}$

$\because$ The tangent // y-axis

$\therefore$ Denominator $= 0$

$\therefore \cos y = 0$

$\therefore y = \dfrac{\pi}{2} + n\pi$ Where $n \in \mathbb{Z}$

, $\because$ in all the choices we find $y \in [0 , \pi]$

$\therefore y = \dfrac{\pi}{2}$

$\therefore 2x + 1 = \sin\dfrac{\pi}{2}$, then $x = 0$

$\therefore$ The point is $\left(0 , \dfrac{\pi}{2}\right)$

10 (b)

Solution :

$\because x = e^t$ $\qquad \therefore \dfrac{dx}{dt} = e^t$

$\because y = e^{2t+2}$ $\qquad \therefore \dfrac{dy}{dt} = 2e^{2t+2}$

$\therefore \dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt}$

at $x = 1$, then $t = 0$

$\therefore$ Slope of tangent at $x = 1$

is $\left(\dfrac{dy}{dx}\right)_{x=1} = 2e^2$

$\therefore$ Slope of normal $= \dfrac{-1}{2e^2}$

11 (b)

Solution :

$x^2 = e^{2y}$ "differentiate with respect to x"

$\therefore 2x = 2e^{2y}\dfrac{dy}{dx}$

$\therefore \dfrac{dy}{dx} = \dfrac{x}{e^{2y}}$ $\qquad \therefore x\dfrac{dy}{dx} = \dfrac{x^2}{e^{2y}} = 1$

12 (a)

Solution :

The figure is a rhombus whose diagonals $2x$, $2y$

$\therefore$ Its area (A) $= \dfrac{1}{2}(2x)(2y) = 2xy$

$\because$ AB = 5 cm. $\qquad \therefore y = \sqrt{25 - x^2}$

$\therefore A = 2x\sqrt{25 - x^2}$

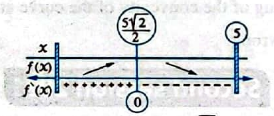
$\therefore \dfrac{dA}{dx} = 2\sqrt{25 - x^2} + 2x\dfrac{-\cancel{2}x}{\cancel{2}\sqrt{25 - x^2}}$

$= \dfrac{2(25 - x^2) - 2x^2}{\sqrt{25 - x^2}} = \dfrac{50 - 4x^2}{\sqrt{25 - x^2}}$

Putting $\dfrac{dA}{dx} = 0$

$\therefore 50 - 4x^2 = 0$ $\qquad \therefore x = \dfrac{5\sqrt{2}}{2}$

and the negative solution is refused



$\therefore$ Area is maximum when $x = \dfrac{5\sqrt{2}}{2}$, $y = \dfrac{5\sqrt{2}}{2}$

i.e. $x = y$

13 (b)

Solution :

$f(x) = x \ln x$

$f'(x) = x \times \dfrac{1}{x} + \ln x = 1 + \ln x$

Putting $f'(x) = 0$

$\therefore \ln x = -1$ $\qquad \therefore x = e^{-1}$

$\therefore f(e) = e$, $f(e^{-1}) = -e^{-1}$, $f(e^{-2}) = -2e^{-2}$

$\therefore$ The absolute maximum is e

14 (b)

Solution :

${}_{-3}\int^{3} \left(f'(x) - f(x)\right) dx$

$= {}_{-3}\int^{3} f'(x)\, dx - {}_{-3}\int^{3} f(x)\, dx$

$= [f(x)]_{-3}^{3} - 2\,{}_{0}\int^{3} f(x)\, dx = f(3) - f(-3) - 2 \times 5$

$= f(3) - f(3) - 10 = -10$ $\qquad$ (The function is even)

15 (c)

Solution :

$y - 1 = e^x$

$\therefore y = e^x + 1$ (doesn't intersect x-axis)

$\therefore$ The required area $= {}_{0}\int^{k} (e^x + 1)\, dx$

$= [e^x + x]_0^k = (e^k + k) - (e^0 + 0)$

$= e^k + k - 1$

16 (a)

Solution :

The side length of the smaller square after time "t"

$= x = 1 + t$

, the side length of the greater square after time "t"

$= y = 4 - \dfrac{1}{2}t$

$\therefore$ The area (A) $= \left(\dfrac{1}{2}\sqrt{2}\,x\right)\left(\dfrac{1}{2}\sqrt{2}\,y\right)$

$- \dfrac{1}{2}x^2$

$= \dfrac{1}{2}(xy - x^2)$

$A = \dfrac{1}{2}\left[(1 + t)\left(4 - \dfrac{1}{2}t\right) - (1 + t)^2\right] = \dfrac{3}{4}(2 + t - t^2)$

$\therefore \dfrac{dA}{dt} = \dfrac{3}{4}(1 - 2t)$

Putting $\dfrac{dA}{dt} = 0$

$\therefore t = \dfrac{1}{2}$

, $\because \dfrac{d^2A}{dt^2} = \dfrac{-3}{2}$ (negative)

$\therefore$ "A" stop increasing instantly at $t = \dfrac{1}{2}$

17 (d)

Solution :

The area of the shaded part

$= {}_{-1}\int^{4} [g(x) - f(x)]\, dx$

$= {}_{-1}\int^{4} g(x)\, dx - {}_{-1}\int^{4} f(x)\, dx$

$= 150 - [(x - 1)^3]_{-1}^{4}$

$= 150 - \left(27 - (-8)\right) = 115$

18 (d)

Solution :

The volume of the solid generated by revolving

$= \pi\, {}_{y=0}\int^{y=a} x^2\, dy = \pi\, {}_{0}\int^{a} (a - y)\, dy$

$= \pi\left[ay - \dfrac{1}{2}y^2\right]_0^a$

$= \pi\left[\dfrac{1}{2}a^2 - 0\right] = \dfrac{1}{2}a^2\pi$

$\therefore \dfrac{1}{2}a^2\pi = 8\pi$ $\qquad \therefore a^2 = 16$

$\therefore a = 4$

19 (a)

Solution :

Equation of the curve $\log_y 4x = 2x$

$\therefore y^{2x} = 4x$ $\qquad \therefore 2x \ln y = \ln 4x$

Differentiate both sides with respect to x

$\therefore 2\ln y + 2x \times \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x}$

At the point (1 , 2) : $2\ln 2 + \dfrac{2 \times 1}{2}\dfrac{dy}{dx} = 1$

$\therefore \dfrac{dy}{dx} = 1 - 2\ln 2$

$\therefore$ The slope of the tangent (m) $= 1 - 2\ln 2$

20 (b)

Solution :

$\because f(x) = 3ax^3 - bx - 5$

$\therefore f'(x) = 9ax^2 - b$

At $x = 1$, f has local maximum value

$\therefore \grave{f}(1) = \text{zero}$ $\qquad\qquad$ $\therefore 9a(1) - b = \text{zero}$

$\therefore \frac{b}{a} = 9$

---

21 (c)

Solution :

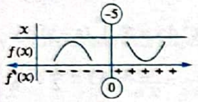
$\because f(x) = e^x (x + 3)$

$\therefore \grave{f}(x) = e^x (x + 3) + e^x = e^x (x + 4)$

$\therefore \grave{\grave{f}}(x) = e^x (x + 4) + e^x = e^x (x + 5)$

Put $\grave{\grave{f}}(x) = 0$:

$\therefore e^x (x + 5) = 0$ $\qquad\qquad$ $\therefore x = -5$



$\therefore$ The curve of the function is convex downward in the interval $]-5 , \infty[$

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22 (d)

Solution :

The curve of the function is convex upward for all values of $x \in \mathbb{R} - \{3\}$ in figure (d)

---

23 (c)

Solution :

$\because \frac{d^2x}{dy^2} = e^{y+1}$ $\qquad$ (integrate with respect to y)

$\therefore \frac{dx}{dy} = \int e^{y+1}\, dy = e^{y+1} + c$

$\because$ The tangent is parallel to the straight line $y = x - 3$

$\therefore$ Its slope $= 1$

$\therefore \frac{dy}{dx} = 1$ hence $\left(\frac{dx}{dy}\right)_{(x , -1)} = 1$

$\therefore 1 = e^{-1+1} + c$ $\qquad\qquad$ $\therefore c = \text{zero}$

$\therefore \frac{dx}{dy} = e^{y+1}$

$\therefore$ The slope of the tangent to the curve at any point is $\frac{dy}{dx} = e^{-y-1}$

24 (c)

Solution :

$\because x = \frac{n}{2}$ $\qquad\qquad$ $\therefore n = 2x$

$\therefore y = (2x) - \frac{(2x)^2}{4} = 2x - x^2$

Put $y = 0$: $\therefore x = 0$ or 2

$\therefore$ The required area $= \int_0^2 (2x - x^2)\, dx$

$= [x^2 - \frac{1}{3} x^3]_0^2$

$= \left(4 - \frac{8}{3}\right) - (0)$

$= \frac{4}{3}$ area units.

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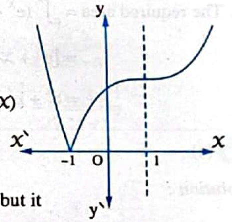
25 (a)

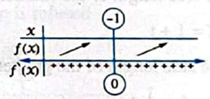
Solution :

$\because \grave{f}(x) = |g(x)|$

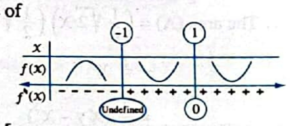
$\therefore$ The opposite figure shows the curve $\grave{f}(x)$ from the figure you notice



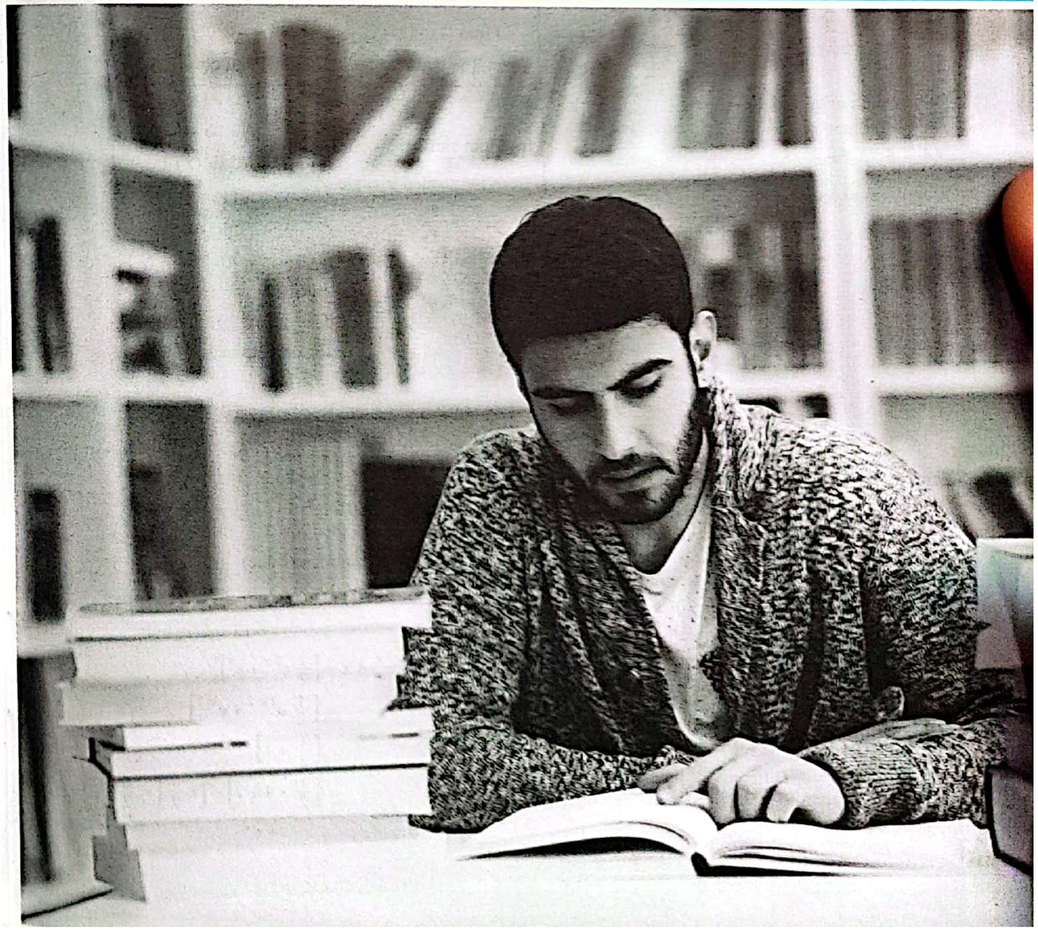
At $x = -1$, $\grave{f}(x) = 0$ but it is not a local minimum



At $x = -1$ only the function f ha an inflection point and the curve of the function is not convex upward in the interval $]-1 , 1[$



Answers of Al-Azhar Exams



Answers of Al-Azhar Exams

First session 2019

1 (1) (a) (2) (d) (3) (a) (4) (c) (5) (b) (6) (d)

2

[a] $x \sin 2y = y \cos 2x$

(by differentiating with respect to x)

$\therefore \sin 2y + 2x \cos 2y \frac{dy}{dx} = \frac{dy}{dx} \cos 2x - 2y \sin 2x$

$\therefore \frac{dy}{dx} = \frac{\sin 2y + 2y \sin 2x}{\cos 2x - 2x \cos 2y}$

$\therefore \left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{4}, \frac{\pi}{2}\right)} = 2$

$\therefore$ slope of tangent $= 2$, slope of normal $= \frac{-1}{2}$

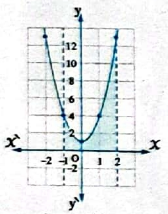
$\therefore$ equation of tangent : $\frac{y - \frac{\pi}{2}}{x - \frac{\pi}{4}} = 2$

i.e. $2x - y = 0$

, equation of normal : $\frac{y - \frac{\pi}{2}}{x - \frac{\pi}{4}} = \frac{-1}{2}$

i.e. $4x + 8y - 5\pi = 0$

[b]



Points of intersection with x-axis :
Put $3x^2 + 1 = 0$

$\therefore$ equation has no solution in $\mathbb{R}$

$\therefore$ The curve doesn't intersect x-axis

$\therefore$ the area $= \int_{-1}^{2} (3x^2 + 1)\, dx$

$= \left[x^3 + x\right]_{-1}^{2} = 12$ square unit.

3

[a] $x^2 y = a \ln x$ (Differentiate with respect to x)

$\therefore x^2 \dot{y} + 2xy = \frac{a}{x}$

$\therefore x^3 \dot{y} + 2x^2 y = a$ (Differentiate with respect to x another time)

$\therefore 3x^2 \dot{y} + x^3 \ddot{y} + 2x^2 \dot{y} + 4xy = 0$

$\therefore 5x^2 \dot{y} + x^3 \ddot{y} + 4xy = 0$ (dividing by x)

$\therefore x^2 \ddot{y} + 5x\dot{y} + 4y = 0$

[b] $\because$ The slope of the normal $= \frac{-1}{a \csc^2 x}$

$\therefore \frac{dy}{dx} = a \csc^2 x$

(by integrating both sides according to x)

$\therefore y = -a \cot x + c$

$\because$ The curve passes through $\left(\frac{\pi}{4}, 3\right), \left(\frac{3\pi}{4}, -1\right)$

$\therefore 3 = -a + c$ (1)

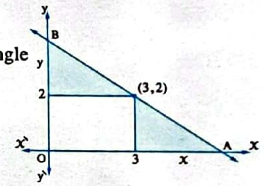
$, -1 = a + c$ (2)

From (1), (2) : $\therefore c = 1$, $a = -2$

$\therefore$ The equation of the curve : $y = 2 \cot x + 1$

4

[a] From similarity of the shaded triangle



$\therefore \frac{x}{3} = \frac{2}{y}$

$\therefore y = \frac{6}{x}$

The area of ΔAOB :

$A = \frac{1}{2}(x + 3)(y + 2)$

$= \frac{1}{2}(x + 3)\left(\frac{6}{x} + 2\right) = 6 + x + 9x^{-1}$

$\therefore \acute{A} = 1 - 9x^{-2}$ $\therefore \grave{A} = 18x^{-3}$

Putting $\acute{A} = 0$ $1 - \frac{9}{x^2} = 0$

$\therefore x^2 = 9$, then $x = 3$

$, \because \grave{A}_{at\ x=3} > 0$ (minimum value)

$\therefore$ The smallest area : $A_{(at\ x=3)} = 6 + 3 + \frac{9}{3}$

$= 12$ square unit.

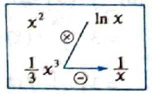
[b] (1) $\int_0^{\ln 2} (e^{2x} + e^x)\, dx = \left[\frac{1}{2} e^{2x} + e^x\right]_0^{\ln 2}$

$= \left[\frac{1}{2} e^{2\ln 2} + e^{\ln 2}\right]$

$- \left[\frac{1}{2} + 1\right]$

$= \left[\frac{1}{2} \times 4 + 2\right] - \left[1\frac{1}{2}\right] = \frac{5}{2}$

(2) $\int x^2 \ln x\, dx$



$= \frac{1}{3} x^3 \ln x$

$- \int \frac{1}{3} x^3 \times \frac{1}{x}\, dx$

$= \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2\, dx$

$= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + c$

5

[a] $f(x) = a x^3 + b x^2$

$\because$ The curve passes through $(1, 12)$

$\therefore f(1) = 12$ $\therefore a + b = 12$ (1)

$, \acute{f}(x) = 3ax^2 + 2bx$

$, \grave{f}(x) = 6ax + 2b$

at $x = 1$ $\therefore \grave{f}(x) = 0$

$\therefore 6a + 2b = 0$ $\therefore 3a + b = 0$ (2)

From (1), (2) :

$\therefore a = -6$, $b = 18$

$\therefore f(x) = -6x^3 + 18x^2$

$, \acute{f}(x) = -18x^2 + 36x$

, at $\acute{f}(x) = 0$

$\therefore -18x(x - 2) = 0$ $\therefore x = 0$ or $x = 2$

$\therefore f(-1) = f(2) = 24$

$, f(0) = f(3) = 0$

$\therefore$ The absolute maximum value is 24 at $x = -1$

$, x = 2$

, The absolute minimum value is 0 at $x = 0$, $x = 3$

[b] (1) $\int \frac{x + 1}{\sqrt{x - 1}}\, dx$ putting $z = x - 1$

$\therefore x = z + 1$ $\therefore dz = dx$

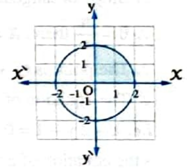
$\therefore \int \frac{x + 1}{\sqrt{x - 1}}\, dx = \int \frac{z + 1 + 1}{\sqrt{z}}\, dz$

$= \int \left((z)^{\frac{1}{2}} + 2z^{\frac{-1}{2}}\right) dz$

$= \frac{(z)^{\frac{3}{2}}}{\frac{3}{2}} + 2 \times \frac{(z)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{2}{3}(x - 1)^{\frac{3}{2}} + 4(x - 1)^{\frac{1}{2}} + c$

(2) $\int_0^2 \sqrt{4 - x^2}\, dx$



Put $y = \sqrt{4 - x^2}$

$\therefore y^2 = 4 - x^2$

$\therefore x^2 + y^2 = 4$

and this is the equation of a circle whose centre is the origin , its radius $= 2$

$\therefore$ The area $= \frac{1}{4}(\pi(2)^2) = \pi$ square unit.

Second session 2019

1 (1) (b) (2) (b) (3) (c) (4) (c) (5) (c) (6) (c)

2

[a] $x = \sec^2\theta - 1$

$\therefore \frac{dx}{d\theta} = 2\sec\theta \cdot \sec\theta\tan\theta = 2\sec^2\theta\tan\theta$

, $y = \tan\theta$ $\qquad \therefore \frac{dy}{d\theta} = \sec^2\theta$

$\therefore \frac{dy}{dx} = \frac{\sec^2\theta}{2\sec^2\theta\tan\theta} = \frac{1}{2\tan\theta}$

$\therefore \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = \frac{1}{2}$

$\therefore$ Slope of tangent $= \frac{1}{2}$, slope of normal $= -2$

$\therefore \theta = \frac{\pi}{4}$ then : $x = 1$, $y = 1$

$\therefore$ The point is $(1, 1)$

$\therefore$ The equation of tangent : $\frac{y-1}{x-1} = \frac{1}{2}$

i.e. $x - 2y + 1 = 0$

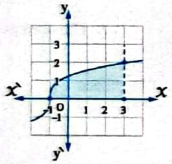
, the equation of normal : $\frac{y-1}{x-1} = -2$

i.e. $2x + y - 3 = 0$

[b] Points of intersection of the curve with

x-axis : put $\sqrt[3]{2x+2} = 0$

$\therefore x = -1$



$\therefore$ The area

$= \int_{-1}^{3} \sqrt[3]{2x+2}\, dx$

$= \left[\frac{(2x+2)^{\frac{4}{3}}}{\frac{4}{3}\times 2}\right]_{-1}^{3} = \left[\frac{3}{8}(2x+2)^{\frac{4}{3}}\right]_{-1}^{3}$

$= \left[\frac{3}{8}(2\times 3+2)^{\frac{4}{3}}\right] - \left[\frac{3}{8}(2\times -1+2)^{\frac{4}{3}}\right]$

$= 6$ square unit.

3

[a] Let the radius length of the circle of the sector $= x$ and the length of the arc $= y$

$\therefore$ The perimeter $= 2x + y = 30$

$\therefore y = 30 - 2x$ $\qquad \therefore$ Its area $(A) = \frac{1}{2}xy$

$\therefore A = \frac{1}{2}x(30 - 2x) = 15x - x^2$

$\therefore A' = 15 - 2x$ $\qquad$, $A'' = -2$

$\therefore A' = 0$ at $x = 7.5$

, $\because A'' = -2 < 0$ (Maximum)

$\therefore$ The radius length of the sector $= 7.5$ cm.

[b] $\int_{-2}^{5} [3f(x) + 6x]\, dx$

$= 3\int_{-2}^{5} f(x)\, dx + [3x^2]_{-2}^{5}$

$= 3\left[\int_{-2}^{3} f(x)\, dx + \int_{3}^{5} f(x)\, dx\right]$

$\quad + [3\times 25 - 3\times 4]$

$= 3[9 + (-4)] + 63 = 78$

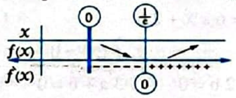
4

[a] $f(x) = x\ln x$, $x > 0$

$\therefore f'(x) = x \times \frac{1}{x} + \ln x = 1 + \ln x$, $f''(x) = \frac{1}{x}$

, $f'(x) = 0$ when $\ln x = -1$

$\therefore x = e^{-1} = \frac{1}{e}$ $\qquad \therefore f\left(\frac{1}{e}\right) = \frac{-1}{e}$



$\therefore$ The function has a local minimum value at $\left(\frac{1}{e}, \frac{-1}{e}\right)$

, $f''(x)$ is undefined at $x = 0 \notin$ domain

$\therefore$ The curve has no inflection point

[b] $\frac{dy}{dx} = 5 - 2\sec^2 2x$

$\therefore y = \int (5 - 2\sec^2 2x)\, dx$

$\therefore y = 5x - \tan 2x + c$

, $\because$ The curve passes through $\left(\frac{\pi}{8}, \frac{5\pi}{8}\right)$

$\therefore \frac{5\pi}{8} = \frac{5\pi}{8} - \tan\frac{\pi}{4} + c$ $\qquad \therefore c = 1$

$\therefore$ The equation of the curve : $y = 5x - \tan 2x + 1$

5

[a] $y = a e^{\frac{b}{x}}$

By taking the natural logarithm of both sides

$\therefore \ln y = \ln a + \frac{b}{x}$

(By differentiating with respect to x)

$\therefore \frac{y'}{y} = \frac{-b}{x^2}$

$\therefore x^2 y' = -by$ $\qquad$ (1)

(By differentiating with respect to x once again)

$\therefore x^2 y'' + 2x y' = -b y'$ $\left(\text{Multiply by } \frac{y}{x}\right)$

$\therefore x y y'' + 2 y y' = \frac{-b y y'}{x}$

From (1) : $x y y'' + 2 y y' = \frac{x^2 (y')^2}{x}$

$\therefore x y y'' + 2 y y' - x (y')^2 = 0$

Another solution :

$\because y = a e^{\frac{b}{x}}$

$\therefore y' = a e^{\frac{b}{x}} \times \frac{-b}{x^2} = \frac{-by}{x^2}$

$\therefore x^2 y' = -by$ "differentiate with respect to x"

$\therefore x^2 y'' + 2x y' = -b y'$

, substitute by "$-b = \frac{x^2 y'}{y}$"

$\therefore x^2 y'' + 2x y' = \frac{x^2 y'^2}{y}$ "Multiply by $\frac{y}{x}$"

$\therefore x y y'' + 2 y y' = x y'^2$

$\therefore x y y'' + 2 y y' - x y'^2 = 0$

[b] (1) $\int 3x\sqrt{2x+3}\, dx$

$= \frac{3}{2}\int 2x\sqrt{2x+3}\, dx$

$= \frac{3}{2}\int (2x + 3 - 3)\sqrt{2x+3}\, dx$

$= \frac{3}{2}\int (2x+3)^{\frac{3}{2}} - 3(2x+3)^{\frac{1}{2}}\, dx$

$= \frac{3}{2}\left[\frac{1}{5}(2x+3)^{\frac{5}{2}} - (2x+3)^{\frac{3}{2}}\right] + c$

$= \frac{3}{10}(2x+3)^{\frac{5}{2}} - \frac{3}{2}(2x+3)^{\frac{3}{2}} + c$

(2) $\int \frac{\ln 5x}{x}\, dx = \frac{(\ln 5x)^2}{2} + c$

First session 2020

1 (1) (a) (2) (c) (3) (c) (4) (a) (5) (b) (6) (a)

2

[a] $x = \sec\theta$ $\quad \therefore \frac{dx}{d\theta} = \sec\theta\tan\theta$

, $y = \tan\theta$ $\quad \therefore \frac{dy}{d\theta} = \sec^2\theta$

$\therefore \frac{dy}{dx} = \frac{\sec^2\theta}{\sec\theta\tan\theta} = \frac{\sec\theta}{\tan\theta} = \csc\theta$

$\therefore \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{3}} = \frac{2}{\sqrt{3}}$

$\therefore$ The slope of tangent $= \frac{2}{\sqrt{3}}$

, the slope of normal $= \frac{-\sqrt{3}}{2}$

at $\theta = \frac{\pi}{3}$ $\quad \therefore x = 2$, $y = \sqrt{3}$

$\therefore$ The point $(2 , \sqrt{3})$

$\therefore$ the equation of tangent : $\frac{y-\sqrt{3}}{x-2} = \frac{2}{\sqrt{3}}$

$\therefore 2x - 4 = \sqrt{3}\,y - 3$

i.e. $2x - \sqrt{3}\,y - 1 = 0$

, the equation of normal : $\frac{y-\sqrt{3}}{x-2} = \frac{-\sqrt{3}}{2}$

$\therefore 2y - 2\sqrt{3} = -\sqrt{3}\,x + 2\sqrt{3}$

i.e. $\sqrt{3}\,x + 2y - 4\sqrt{3} = 0$

[b] $f(x) = |x-3| = \begin{cases} -x+3 & , x < 3 \\ x-3 & , x \geq 3 \end{cases}$

$\therefore \int_0^5 f(x)\,dx = \int_0^3 (-x+3)\,dx$

$+ \int_3^5 (x-3)\,dx$

$= \left[\frac{-1}{2}x^2 + 3x\right]_0^3$

$+ \left[\frac{1}{2}x^2 - 3x\right]_3^5$

$= \frac{9}{2} + 2 = \frac{13}{2}$

3

[a] The slope of the normal $= (4y+3)\sec x$

$\therefore$ The slope of the tangent $= \frac{dy}{dx} = \frac{-1}{(4y+3)\sec x}$

$\therefore \int (4y+3)\,dy = -\int \cos x\,dx$

$\therefore 2y^2 + 3y = -\sin x + c$

, $\because$ the curve passes throught (0 , 0)

$\therefore c = 0$

$\therefore$ The equation of the curve is :

$2y^2 + 3y + \sin x = 0$

[b] $\because y' = a\,e^{\frac{b}{x}} \times \frac{-b}{x^2} = \frac{-b\,y}{x^2}$

$\therefore x^2 y' = -b\,y$ "differentiate with respect to x"

$\therefore x^2 y'' + 2x y' = -b\,y'$

, substitute by "$-b = \frac{x^2 y'}{y}$"

$\therefore x^2 y'' + 2x y' = \frac{x^2 y'^2}{y}$ "Multiply by $\frac{y}{x}$"

$\therefore x y y'' + 2y y' = x y'^2$

$\therefore x y y'' + 2y y' - x y'^2 = 0$

Another solution :

$y = a\,e^{\frac{b}{x}}$ (by taking natural logarithm to both sides)

$\therefore \ln y = \ln a + \frac{b}{x}$

(By differentiate with respect to x)

$\therefore \frac{y'}{y} = \frac{-b}{x^2}$

$\therefore x^2 y' = -b\,y$ (1)

(by differentiate once again with respect to x)

$x^2 y'' + 2x y' = -b\,y'$ (Multiply by y)

$\therefore x^2 y y'' + 2x y y' = -b\,y\,y'$

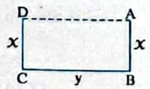
From (1) :

$\therefore x^2 y y'' + 2x y y' = x^2 y' \times y'$ (divide by x)

$\therefore x y y'' + 2y y' - x (y')^2 = 0$

4

[a] Let the dimensions of the rectangle are x , y



$\therefore 2x + y = 800$ metre

$\therefore y = 800 - 2x$

Let the area of the rectangle (A) $= x\,y$

$\therefore A = x(800 - 2x) = 800x - 2x^2$

$\therefore A' = 800 - 4x$

Putting $A' = 0$ $\quad \therefore x = 200$ m.

, $\because A'' = -4 < 0$

$\therefore$ A has maximum value at $x = 200$ m.

$\therefore$ The dimensions of the rectangle 200 m. , 400 m.

$\therefore$ The largest area $= 200 \times 400 = 80000\ m^2$.

[b] $\because \int_1^4 g(x)\,dx = -3$

$\therefore \int_1^4 [f(x) + 2g(x) - 4]$

$= \int_1^4 f(x)\,dx + 2\int_1^4 g(x)\,dx - \int_1^4 4\,dx$

$= 7 + 2 \times -3 - [4x]_1^4$

$= 7 - 6 - (16 - 4) = -11$

5

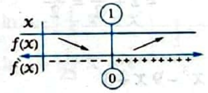
[a] $\because f(x) = (x-1)^4 + 3$

$f'(x) = 4(x-1)^3$, $f''(x) = 12(x-1)^2$

Putting $f'(x) = 0$

$\therefore 4(x-1)^3 = 0$

$\therefore x = 1$



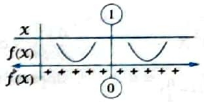
$\therefore f$ is decreasing on $]-\infty , 1[$, increasing on $]1 , \infty[$

, The function has a local minimum value at $x = 1$, $f(1) = 3$

, putting $f''(x) = 0$

$\therefore 12(x-1)^2 = 0$

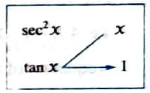
$\therefore x = 1$



The curve is convex downwards at $]-\infty , 1[$, $]1 , \infty[$

no inflection points that the curve has.

[b] (1) $\int x\sec^2 x\,dx$



$= x\tan x - \int \tan x\,dx$

$= x\tan x - \int \frac{\sin x}{\cos x}\,dx$

$= x\tan x + \ln|\cos x| + c$

(2) $\int x\sqrt[3]{x-1}\,dx$

$= \int (x-1+1)\sqrt[3]{x-1}\,dx$

$= \int \left((x-1)^{\frac{4}{3}} + (x-1)^{\frac{1}{3}}\right)dx$

$= \frac{3}{7}(x-1)^{\frac{7}{3}} + \frac{3}{4}(x-1)^{\frac{4}{3}} + c$

Second session 2020

1 (1) (d) (2) (a) (3) (a) (4) (b) (5) (b) (6) (d)

2

[a] $y = 3 - \cot^2 x$ $\qquad \therefore \frac{dy}{dx} = 2 \cot x \csc^2 x$

$\therefore \left(\frac{dy}{dx}\right)_{(x=\frac{\pi}{4})} = 4$ $\qquad \therefore$ slope of tangent $= 4$

, slope of normal $= \frac{-1}{4}$

, when $x = \frac{\pi}{4}$ $\qquad \therefore y = 2$

the point $\left(\frac{\pi}{4}, 2\right)$

$\therefore$ equation of tangent : $\frac{y-2}{x-\frac{\pi}{4}} = 4$

i.e. $4x - y - \pi + 2 = 0$

, equation of normal : $\frac{y-2}{x-\frac{\pi}{4}} = \frac{-1}{4}$

i.e. $4x + 16y - \pi - 32 = 0$

[b] (1) $\int_0^1 \frac{3e^x - 2e^{2x}}{2e^x} dx$

$= \int_0^1 \left(\frac{3}{2} - e^x\right) dx$

$= \left[\frac{3}{2}x - e^x\right]_0^1 = \left(\frac{3}{2} - e\right) - (-1) = \frac{5}{2} - e$

(2) $\int \frac{(3x-1)^2}{3x} dx = \int \frac{9x^2 - 6x + 1}{3x} dx$

$= \int \left(3x - 2 + \frac{1}{3x}\right) dx$

$= \frac{3}{2}x^2 - 2x + \frac{1}{3} \ln |x| + c$

3

[a] $y^2 + 2xy = 8$ (by differentiate with respect to x)

$\therefore 2y \frac{dy}{dx} + 2y + 2x \frac{dy}{dx} = 0$ (divide by 2)

$\therefore \frac{dy}{dx}(x + y) + y = 0$

(by differentiate once more with respect to x)

$\therefore \frac{d^2y}{dx^2}(x+y) + \frac{dy}{dx}\left(1 + \frac{dy}{dx}\right) + \frac{dy}{dx} = 0$

$\therefore (x+y)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 = 0$

[b] $\frac{dy}{dx} = \frac{2x+3}{x} = 2 + \frac{3}{x}$

(by integration both sides with respect to x)

$y = \int \left(2 + \frac{3}{x}\right) dx = 2x + 3 \ln|x| + c$

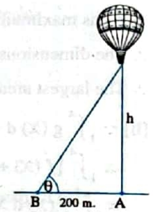
$\because (e, 2e+5)$ lies on the curve

$\therefore 2e + 5 = 2e + 3 + c$ $\qquad \therefore c = 2$

$\therefore y = 2x + 3 \ln|x| + 2$

4

[a] $\frac{d\theta}{dt} = 0.12$ rad./min.



, $\tan \theta = \frac{h}{200}$ $\qquad \therefore h = 200 \tan \theta$

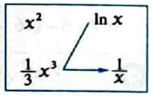
$\therefore \frac{dh}{dt} = 200 \sec^2 \theta \frac{d\theta}{dt}$

when : $\theta = \frac{\pi}{4}$

$\therefore \frac{dh}{dt} = 200 \sec^2 \frac{\pi}{4} \times 0.12$

$= 48$ m./min.

[b] (1) $\int x^2 \ln x \, dx$



$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$

$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c$

(2) $f(x) = |x - 2| = \begin{cases} -x+2, & x < 2 \\ x - 2, & x \geq 2 \end{cases}$

$\therefore \int_0^5 |x-2| dx = \int_0^2 (-x+2) dx + \int_2^5 (x-2) dx$

$= \left[-\frac{1}{2}x^2 + 2x\right]_0^2 + \left[\frac{1}{2}x^2 - 2x\right]_2^5$

$= 2 + \frac{9}{2} = \frac{13}{2}$

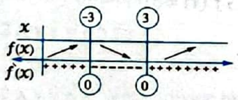
5

[a] $f(x) = \frac{1}{3}x^3 - 9x + 3$

$f'(x) = x^2 - 9$

, putting $f'(x) = 0$

$\therefore x^2 - 9 = 0$ $\qquad \therefore x = \pm 3$



f is decreasing on $]-3, 3[$

, increasing on $]-\infty, -3[$, $]3, \infty[$

, The function has a local maximum value at $x = -3$, $f(-3) = 21$

, the function has a local minimum value at $x = 3$, $f(3) = -15$

[b] (1) $\int \frac{\cos^3 x - 5}{1 - \sin^2 x} dx = \int \frac{\cos^3 x - 5}{\cos^2 x} dx$

$= \int (\cos x - 5 \sec^2 x) dx = \sin x - 5 \tan x + c$

(2) $\int \frac{\ln 5x}{x} dx = \int \frac{1}{x} (\ln 5x) dx = \frac{(\ln 5x)^2}{2} + c$

First session 2021

1 (1) (d) (2) (a) (3) (b) (4) (d) (5) (c) (6) (a)

2

[a] $y = 3 - \cot^2 x$ $\qquad \therefore \frac{dy}{dx} = 2 \cot x \csc^2 x$

$\therefore \left(\frac{dy}{dx}\right)_{(x=\frac{\pi}{4})} = 4$ $\qquad \therefore$ Slope of tangent $= 4$

, slope of normal $= \frac{-1}{4}$

, when $x = \frac{\pi}{4}$ $\qquad \therefore y = 2$

The point $\left(\frac{\pi}{4}, 2\right)$

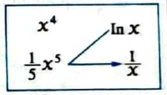
$\therefore$ Equation of tangent : $\frac{y-2}{x-\frac{\pi}{4}} = 4$

i.e. $4x - y - \pi + 2 = 0$

, equation of normal : $\frac{y-2}{x-\frac{\pi}{4}} = \frac{-1}{4}$

i.e. $4x + 16y - \pi - 32 = 0$

[b] (1) $\int x^4 \ln x \, dx$



$= \frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^4 dx$

$= \frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 + c$

(2) $\int x^2 \sqrt{x-3} \, dx$

Put $z = \sqrt{x-3}$

$\therefore x = z^2 + 3$ $\qquad \therefore dx = 2z \, dz$

$\therefore \int x^2 \sqrt{x-3} \, dx = \int (z^2+3)^2 \times z \times 2z \, dz$

$= \int 2z^2 (z^4 + 6z^2 + 9) dz$

$= \int (2z^6 + 12z^4 + 18z^2) dz$

$= \frac{2}{7}z^7 + \frac{12}{5}z^5 + 6z^3 + c$

$= \frac{2}{7}(x-3)^{\frac{7}{2}} + \frac{12}{5}(x-3)^{\frac{5}{2}} + 6(x-3)^{\frac{3}{2}} + c$

3

[a] $\sin y + \cos 2x = 0$ "Differentiate with respect to x"

$\therefore \cos y \frac{dy}{dx} - 2 \sin 2x = 0$

Differentiate with respect to x another time.

$\therefore \cos y \frac{d^2y}{dx^2} + \frac{dy}{dx} \times - \sin y \times \frac{dy}{dx} - 4 \cos 2x = 0$

(dividing by cos y)

$\therefore \frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 \tan y = 4 \cos 2x \sec y$

[b] Slope of the tangent $= \frac{dy}{dx} = 3 + \frac{2}{x}$

$\therefore y = \int 3 + \frac{2}{x} dx = 3x + 2 \ln|x| + c$

, $\because$ the curve passes through $(e, 3e + 5)$

$\therefore 3e + 5 = 3e + 2 \ln e + c$ $\qquad \therefore c = 3$

$\therefore y = 3x + 2 \ln|x| + 3$

4

[a] Let the base side length $= x$, height $= h$

$\therefore 8x + 4h = 120$ $\qquad 2x + h = 30$

$\therefore h = 30 - 2x$ $\qquad$ (1)

$v = x^2 h$

From (1) : $v = x^2(30 - 2x) = 30x^2 - 2x^3$

$\therefore v' = 60x - 6x^2$

Put $v' = 0$: $\therefore x = 0$ (Refused) or $x = 10$

, $\because v' = 60x - 6x^2$

$\therefore v'' = 60 - 12x$

At $x = 10$: $v'' = -60 < 0$

$\therefore$ There is a local maximum

$\therefore$ At $x = 10$ cm. the volume is maximum

From (1) : h = 10 cm.

$\therefore$ The dimensions of the box that has greatest volume are 10 , 10 , 10 cm.

[b] $\int_{-2}^{5} [3f(x) - 6] dx$

$= 3\int_{-2}^{5} f(x) dx - [6x]_{-2}^{5}$

$= 3\left[\int_{-2}^{3} f(x) dx + \int_{3}^{5} f(x) dx\right]$

$- [30 - (-12)] = 3(9 + (-4)) - 42 = -27$

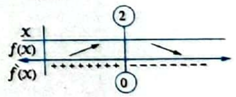
5

[a] (1) $f(x) = e^x (3 - x)$

$\therefore f'(x) = e^x \times -1 + e^x (3 - x) = 2e^x - x e^x$

$= (2 - x) e^x$

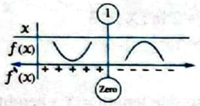
$\therefore f'(x) = 0$ at $x = 2$



The function has a local maximum value $f(2) = e^2$

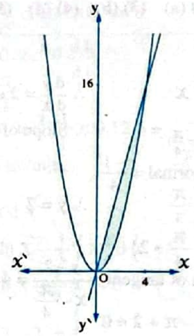
(2) $f''(x) = (2 - x)e^x - e^x = e^x(1 - x)$

, $f''(x) = 0$ at $x = 1$



$\therefore$ The curve is convex upward in $]1, \infty[$ and convex downward in $]-\infty, 1[$ and has an inflection point at $(1, 2e)$

[b] By solving the two equations the points of intersection at $x = 0$, $x = 4$



$\therefore$ The area $= \int_0^4 (4x - x^2)\,dx$

$= \left[2x^2 - \frac{1}{3}x^3\right]_0^4$

$= \frac{32}{3}$ square unis.

Second session 2021

1 (1) (c) (2) (a) (3) (a) (4) (c) (5) (b) (6) (d)

2

[a] $y = \sec x$ $\quad \therefore \frac{dy}{dx} = \sec x \tan x$

$\therefore \frac{d^2y}{dx^2} = \sec x \sec^2 x + \sec x \tan x \tan x$

$= \sec^3 x + \sec x \tan^2 x$

$= \sec x (\sec^2 x + \tan^2 x)$

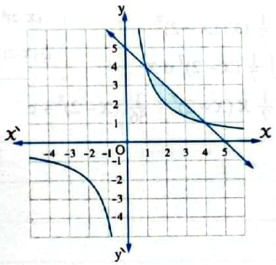
$\therefore y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \sec x \,.\, \sec x (\sec^2 x + \tan^2 x) + \sec^2 x \tan^2 x$

$= \sec^2 x (\sec^2 x + 2\tan^2 x)$

$= \sec^2 x (3\sec^2 x - 2)$

$= y^2(3y^2 - 2)$

[b]



$y_1 = \frac{4}{x}$, $y_2 = 5 - x$

Intersecting at $x = 1$, $x = 4$

$\because y_1 \le y_2$ at the interval $[1, 4]$

$\therefore$ The volume $= \pi \int_1^4 (y_2^2 - y_1^2)\,dx$

$= \pi \int_1^4 (5 - x)^2 - \left(\frac{4}{x}\right)^2 dx$

$= \pi \int_1^4 \left(25 - 10x + x^2 - \frac{16}{x^2}\right) dx$

$= \pi \left[25x - 5x^2 + \frac{1}{3}x^3 + 16x^{-1}\right]_1^4$

$= 9\pi$ cubic unit.

3

[a] Let the volume of the balloon "v" and the surface area of the balloon "A"

(1) $v = \frac{4}{3}\pi r^3$

$\therefore \frac{dv}{dr} = 4\pi r^2 \times \frac{dr}{dt}$

$\therefore 8\pi = 4\pi \times (4)^2 \times \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{1}{8}$ cm./sec.

(2) $A = 4\pi r^2$ $\quad \frac{dA}{dt} = 8\pi r \frac{dr}{dt}$

$\therefore \frac{dA}{dt} = 8\pi \times 4 \times \frac{1}{8} = 4\pi$ cm²/sec.

[b] $\int_1^4 (f(x) + 2g(x) - 4)\,dx$

$= \int_1^4 f(x)\,dx + 2\int_1^4 g(x)\,dx - 4\int_1^4 dx$

$= 7 + 2 \times -3 - 4[x]_1^4$

$= 7 + (-6) - 4[4 - 1] = -11$

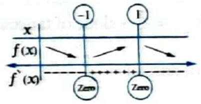
4

[a] $f(x) = 5 + 3x - x^3$

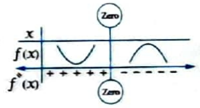
$\therefore f'(x) = 3 - 3x^2$

, $f''(x) = -6x$

Put $f'(x) = 0$: $\therefore x = \pm 1$



$\therefore$ There is a local minimum $f(-1) = 3$ and local maximum $f(1) = 7$

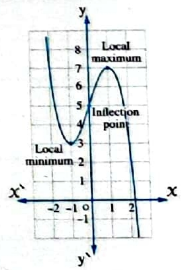


Put $f''(x) = 0$: $\therefore x = 0$

$\therefore$ The point of inflection $(0, 5)$

Assisstant points $f(-2) = 7$, $f(2) = 3$

x	-2	-1	0	1	2
$f(x)$	7	3	5	7	3



[b] $\frac{dy}{dx} = 7 - 2e^x$

(By integration both sides with respect to x)

$\therefore y = \int (7 - 2e^x)\, dx = 7x - 2e^x + c$

$\because f(\ln 2) = 3$

$\therefore 3 = 7 \ln 2 - 2e^{\ln 2} + c$

$\therefore 3 = 7 \ln 2 - 2 \times 2 + c$

$\therefore c = 7 - 7 \ln 2 = 7(1 - \ln 2) = 7\left(\ln \frac{e}{2}\right)$

$\therefore f(x) = 7x - 2e^x + 7 \ln \frac{e}{2}$

5

[a] $x = t^2 + 4t$ $\qquad \therefore \frac{dx}{dt} = 2t + 4$

, $y = 2t^2$ $\qquad \therefore \frac{dy}{dt} = 4t$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{4t}{2t+4}$

$\therefore \left(\frac{dy}{dx}\right)_{t=1} = \frac{2}{3}$, slope of tangent $= \frac{2}{3}$

$\therefore$ Slope of normal $= \frac{-3}{2}$

, when $t = 1$ $\qquad \therefore x = 5$, $y = 2$

$\therefore$ Equation of tangent is :

$= \frac{y-2}{x-5} = \frac{2}{3}$

i.e. $3y - 2x + 4 = 0$

, equation of normal is $\frac{y-2}{x-5} = \frac{-3}{2}$

i.e. $2y + 3x - 19 = 0$

[b] (1) $\int 2x \cos(2x - 1)\, dx$

$$\begin{array}{ll} 2\cos(2x-1) & x \\ \sin(2x-1) & 1 \end{array}$$

$= x \sin(2x - 1)$

$- \int \sin(2x - 1)\, dx$

$= x \sin(2x - 1) + \frac{1}{2} \cos(2x - 1) + c$

(2) $\int x (x-2)^6\, dx$

$$\begin{array}{ll} (x-2)^6 & x \\ \frac{1}{7}(x-2)^7 & 1 \end{array}$$

$= \frac{1}{7} (x) (x-2)^7$

$- \int \frac{1}{7} (x-2)^7\, dx$

$= \frac{1}{7} x (x-2)^7 - \frac{1}{56} (x-2)^8 + c$

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